Numerical Analysis Qualifiers 2007, 2006, 2005, 2004, and some 2003

Notes:

- These are generally really rough ideas of how I might go about these problems.
- I do not guarantee the solutions are even close to correct, but they are the first thing I could think of. Hopefully they'll get you thinking in maybe a different way then you've already tried if you're stuck on one of these problems.
- Don't be surprised at all if the things I suggest lead to a dead end.
- The numbering scheme here is day.problem where day is the day of the exam (1 or 2) and problem is the problem number. I attempted to catch all of what you might consider purely numerical analysis questions (some are closer to what I would call applied analysis, but they still have some numerical component to them), but I may have forgotten a few or done a few other problems by mistake.
- It appears problems were mislabeled in the exams. I recieved a lot of loose sheets, so some problems here may not correspond to the right exams.
- The solutions get a bit more sparse as these go on. This is simply because some of the same type of problems pop up over and over again and I didn't want to write full solutions that do the same thing as a previous problem.

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1.4. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, and that $y \in \mathbb{R}^n$with $F(y) = y$. Further suppose that we know that all of the eigenvalues λ of $F'(y)$ satisfy $\left|\lambda^2 - \lambda - 6\right| < 1$. Show that the iteration defined by $x_{n+1} = -\left(F\left(F\left(x_n\right)\right) - F\left(x_n\right) - 6x_n\right)/6$ will converge to y if x_0 is close enough to y. Estimate the rate of convergence.

Well, we know that there is a point $y \in \mathbb{R}^n$with $F(y) = y$, thus F has a fixed point. Let us rewrite our iteration as

$$x_{n+1} = G\left(x_n\right).$$

We wish to show this iteration converges to y if x_0 is close to y. My first guess would be some sort of Mean Value Theorem, so

$$x_{n+1} - y = G\left(x_n\right) - G\left(y\right) = G'\left(\xi\right)\left(x_n - y\right)$$

and show that, in some norm, $G'\left(\xi\right) \leq 1$. The rate of convergence can be estimated simply by the magnitude of $G'(\xi)$.

Alternatively, you may be able to attempt some sort of analysis straight by some sort of Cauchy criterion. Perhaps write the iteration as

$$x_{n+1} - x_n = \frac{F\left(x_n\right) - F\left(F\left(x_n\right)\right)}{6}$$

slap some norms on there and see what happens.

1.5 Find the SVD of the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

No need to write much explanation here. Just do the problem (if you don't know what SVD is you're screwed no matter what). Recall the SVD is the Singular Value Decomposition, which means we wish to write the matrix A as $A = U\Sigma V^T$ (here $V^* = V^T$) where U and V are unitary and Σ is a diagonal matrix. Recall that, for a general matrix, the method of finding the SVD is as follows:

- Compute A^TA
- Find eigenvalues of A^TA by solving $\det\left(A^TA - \lambda I\right)$. The positive square root of the nonzero eigenvalues become your singular values, so $\sigma_i = \sqrt{\lambda_i}$ for $\lambda_i \neq 0$.
- The V matrix ends up being the normalized eigenvectors of A^TA corresponding to the values in Σ, in order (so, normalize these vectors, they should already be orthogonal since A^TA was SPD).
- The U matrix is found by making each column of U: $u_i = \dfrac{Av_i}{\sigma_i}$

For our problem, we've got

$$A^TA = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is awfully nice. Clearly our eigenvalues are 9 and 1. Now, we find the eigenvectors. I believe it is pretty clear that the eigenvector for 9 is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and the eigenvectors for 1 are $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Now

$$u_1 = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} /\sqrt{9} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; u_2 = A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} /\sqrt{1} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \; u_3 = A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} /\sqrt{1} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

and we get

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, if we had been a little brighter, we should have noticed that the columns of A were already orthogonal, and nearly orthonormal, so simply dividing out the normalizing factor would have given us the SVD.

1.6 Given the (4×2) matrix A below, find the pseudoinverse A^+ and a (4×4) projector P that projects a vector onto the subspace spanned by the columns of A.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

What is a pseudoinverse? Well $A^+ = (A^*A)^{-1} A^*$. When is this used? In the context of least squares problems, where we wish to compute $x = A^+b$ and $y = Pb$ (when minimizing $r = b - Ax$). In other words, this is the pseudoinverse because it solves $A^*Ax = A^*b$ (or $Pb = Ax$). One way to do this is to use SVD like in the last problem. Then $P = UU^*$ and $A^+ = V\Sigma^{-1}U^*$. Frankly, I would just do this, so you ONLY have to remember the SVD and recalling what a pseudoinverse is comes along with it. If you don't remember this, I'd try to go straight from the definition. Why? Well, A is 4×2 so A^*A is just 2×2 so I won't have a problem with computing matrix inverses or anything like that so A^+ will be very easy to find (note that P will not be obvious when doing it like this). We have

$$A^*A = \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix}, \; (A^*A)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & \frac{4}{3} \end{pmatrix}$$

and

$$A^+ = (A^*A)^{-1} A^* = \begin{pmatrix} 1 & -1 \\ -1 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 \end{pmatrix}.$$

So you can quickly get A^+.

Now, for finding P (again, without using SVD!), you'd have to do a bit of thinking. Since $Pb = Ax$ and $x = A^+b$ then for $b = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix}^T$ we'd have

$$x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} b_4 \\ \frac{b_1}{3} + \frac{b_2}{3} + \frac{b_3}{3} - b_4 \end{pmatrix}$$

and

$$Ax = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_4 \\ \frac{b_1}{3} + \frac{b_2}{3} + \frac{b_3}{3} - b_4 \end{pmatrix} = \begin{pmatrix} \frac{b_1}{3} + \frac{b_2}{3} + \frac{b_3}{3} \\ \frac{b_1}{3} + \frac{b_2}{3} + \frac{b_3}{3} \\ \frac{b_1}{3} + \frac{b_2}{3} + \frac{b_3}{3} \\ b_4 \end{pmatrix} = P \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

This seems a bit odd, but I believe it works. Recall that if P is a projection then $P^2 = P$ and $Px = x$ for each x that are basis vectors for the range of A. Well, we have $P^2 = P$ and

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

2.1 Suppose that P and Q are both $n \times n$ matrices which represent orthogonal projections. I.e. $P = P^T$, $P^2 = P$, $Q = Q^T$, $Q^2 = Q$. Let $A = PQ$. Show that the spectral radius of A is ≤ 1.

So we need to show $\rho(A) = \rho(PQ) \leq 1$. Recall that $\rho(A) \leq \|A\|$ for any induced norm. Thus

$$\rho(A) \leq \|PQ\|_2 \leq \|P\|_2 \|Q\|_2 \leq 1.$$

The last inequality comes because the norm of any projection matrix is 1. For example, $\|P\|_2 = \|P^2\|_2 \leq \|P\|_2 \|P\|_2$ which implies $\|P\|_2 \leq 1$. Note that I chose the L^2 norm here because you also get the little trick of

$$\|P\|_2 = \sqrt{\rho(PP^T)} = \rho(P)$$

which can sometimes be useful.

2.3 By differentiating the polynomial passing through y_{n-2}, y_{n-1}, y_n obtain the coefficients of the implicit multistep method in the form

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} = h\beta_0 f_n$$

assuming uniform step h. Normalize coefficients to make $\alpha_0 = 1$.

Recall that a multistep method, as There seems to be quite a few ways to do this correctly, but we'll try to go straight from the suggestion. What is the polynomial passing through these points? The simplest way to do this is to to use standard Lagrange polynomials. We have

$$\begin{aligned} P(x) &= y_{n-2}\frac{x-x_n}{x_{n-2}-x_n}\frac{x-x_{n-1}}{x_{n-2}-x_{n-1}} + y_{n-1}\frac{x-x_{n-2}}{x_{n-1}-x_{n-2}}\frac{x-x_n}{x_{n-1}-x_n} + y_n\frac{x-x_{n-2}}{x_n-x_{n-2}}\frac{x-x_{n-1}}{x_n-x_{n-1}} \\ &= y_{n-2}\frac{1}{2h^2}(x-x_n)(x-x_{n-1}) + y_{n-1}\frac{-1}{h^2}(x-x_n)(x-x_{n-2}) + y_n\frac{1}{2h^2}(x-x_{n-1})(x-x_{n-2}). \end{aligned}$$

Note that

$$\begin{aligned} P(x_{n-2}) &= y_{n-2}\frac{1}{2h^2}(x_{n-2}-x_n)(x_{n-2}-x_{n-1}) = y_{n-2}\frac{1}{2h^2}(-2h)(-h) = y_{n-2} \\ P(x_{n-1}) &= y_{n-1}\frac{-1}{h^2}(x_{n-1}-x_n)(x_{n-1}-x_{n-2}) = y_{n-1}\frac{-1}{h^2}(-h)(h) = y_{n-1} \\ P(x_n) &= y_n\frac{1}{2h^2}(x_n-x_{n-1})(x_n-x_{n-2}) = y_n\frac{1}{2h^2}(h)(2h) = y_n. \end{aligned}$$

The great thing about using Lagrange polynomials is that we now have

$$P'(x_{n-2}) = y'(x_{n-2}) = f(x_{n-2}, y_{n-2}), \text{ etc.}$$

Now we just toss this into our multistep method

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} = \int_{x_{n-2}}^{x_n} f(x)\,dx \approx \int_{x_{n-2}}^{x_n} P'(x)\,dx.$$

So we need to approximate $\int_{x_{n-2}}^{x_n} P'(x)\,dx$. We have

$$\begin{aligned} P'(x) &= y_{n-2}\frac{1}{2h^2}[2x-x_n-x_{n-1}] + y_{n-1}\frac{-1}{h^2}[2x-x_n-x_{n-2}] + y_n\frac{1}{2h^2}[2x-x_{n-1}-x_{n-2}] \\ &= \frac{1}{2h^2}[y_{n-2}(-x_n-x_{n-1}) + 2y_{n-1}(x_n+x_{n-2}) + y_n(-x_{n-2}-x_{n-1})] \\ &\quad + \frac{x}{h^2}[y_{n-2} - 2y_{n-1} + y_n] \end{aligned}$$

so

$$\int_{x_{n-2}}^{x_n} P'(x)\,dx = \frac{1}{2h^2}\left[y_{n-2}\left(-x_n - x_{n-1}\right) + 2y_{n-1}\left(x_n + x_{n-2}\right) + y_n\left(-x_{n-2} - x_{n-1}\right)\right]\left(x_n - x_{n-2}\right) + \frac{\left(x_n - x_{n-2}\right)^2}{2h^2}\left[y_{n-2} - 2y_{n-1} + y_n\right].$$

Assume $h = 1$ and our interval is 0 to 2 (this could have been done in a previous step). Then

$$\int_{x_{n-2}}^{x_n} P'(x)\,dx = \tfrac{1}{2}\left[y_{n-2}\left(-3\right) + 2y_{n-1}\left(2\right) + y_n\left(-1\right)\right] + 2\left[y_{n-2} - 2y_{n-1} + y_n\right] = \frac{1}{2}y_{n-2} - 2y_{n-1} + \frac{3}{2}y_n.$$

Now we have

$$\frac{3}{2}y_n - 2y_{n-1} + \frac{1}{2}y_{n-2} = h\beta_0 f_n.$$

Dividing by $\frac{3}{2}$ we have

$$y_n - \frac{4}{3}y_{n-1} + \frac{1}{3}y_{n-2} = h\beta_0 f_n$$

(where I just incorporated the 3/2 into the β_0 constant).

All that is left is to find β_0. We do this using our Taylor series type methods. Recall for 0^{th} order we need

$$\sum_{i=0}^{2} a_i = 0 \rightarrow \alpha_{n-2} + \alpha_{n-1} + \alpha_n = \frac{1}{3} - \frac{4}{3} + 1 = 0.$$

For 1^{st} order we need

$$\sum_{i=0}^{2}\left(ia_i - b_i\right) = 0 \rightarrow 1\cdot\alpha_{n-1} + 2\cdot\alpha_n - \beta_0 = -\frac{4}{3} + 2 - \beta_0 = 0$$

thus $\beta_0 = \frac{2}{3}$. (Note that this process continues, for 2^{nd} order we need $\sum_{i=0}^{2}\left(\frac{1}{2}ia_i^2 - ib_i\right) = 0$ and so on). Finally, we have our multistep method

$$y_n - \frac{4}{3}y_{n-1} + \frac{1}{3}y_{n-2} = h\frac{2}{3}\beta_0 f_n.$$

2.8 (a) Is the linear multistep method

$$y_n = -4y_{n-1} + 5y_{n-2} + 4hf_{n-1} + 2hf_{n-2}$$

stable? Explain why or why not.

(b) Prove that the method

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \alpha_3 y_{n-3} = h\beta_0 f_n$$

with $\beta_0 = \frac{6}{11}$, $\alpha_0 = 1$, $\alpha_1 = -\frac{18}{11}$, $\alpha_2 = \frac{9}{11}$, $\alpha_3 = -\frac{2}{11}$ has stiff decay.

Starting at (a) we attempt to show stability. What is stability? It means all of the roots of the polynomial formed by the a coefficients (coefficients of y and not f) have magnitude less than 1 and simple roots if they have modulus 1 We have $y_n = -4y_{n-1} + 5y_{n-2} + 4hf_{n-1} + 2hf_{n-2}$ or $y_n + 4y_{n-1} - 5y_{n-2} = 4hf_{n-1} + 2hf_{n-2}$. This multistep runs from $n-2$ to n thus our polynomial for the y terms is

$$p(z) = z^2 + 4z - 5 = \left(z + 5\right)\left(z - 1\right).$$

this is most definitely not stable. A nice way to see this is to consider $y' = 0$. Then we have

$$y_n + 4y_{n-1} - 5y_{n-2} = 0$$

which has solutions $y_n = h\lambda^n$ where λ is a root of p. Since $\lambda = -5$ we'd have $y_n = h(-5)^n \rightarrow \infty$.

As for (b), I'm not 100% sure what they mean by stiff decay. I believe they mean that you are to show for $y' = \lambda y - \lambda g(x)$ that $|y_n - g(x_n)| \rightarrow 0$ as $kRe\left(\lambda\right) \rightarrow -\infty$. I do not believe this is something very standard, and it is not part of any of Juan's notes (or even many of the standard numerical texts) so we will ignore this for now. If you all feel it is very important, I'll take another look at it.

January 2007

1.3 For the linear system of equations in the unknowns x, y, z

$$\begin{aligned} 6x - 2y + 3kz &= 3 \\ 6x + y &= 2k \\ 2kx + 3z &= 3 \end{aligned}$$

Determine the values of k for which the system has:

(a) no solution, (b) a unique solution, (c) many solutions.

Well, this is pretty straightforward. So essentially, when is this overdetermined, under determined and determined (is that a term?). You can do this as a matrix, but truthfully it might be easier just straight from the system to start with. Solve the second equation for y and get $y = 2k - 6x$ and plug into the first to get

$$\begin{aligned} 18x + 3kz &= 3 + 4k \\ 2kx + 3z &= 3. \end{aligned}$$

Now, as a matrix we'd have

$$\begin{pmatrix} 18 & 3k \\ 2k & 3 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 3 + 4k \\ 4 \end{pmatrix}.$$

Take the determinant of the matrix to get $54 - 6k^2$. This is zero only if $k^2 = 9$ or $k = \pm 3$. Let us see what happens in each of these cases.

For $k = 3$ we have

$$\begin{aligned} 6x - 2y + 9z &= 3 \\ 6x + y &= 6 \\ 6x + 3z &= 3. \end{aligned}$$

Solving for z we have $z = 1 - 2x$ and solving for y we have $y = 6 - 6x$ and putting into the 1st we have

$$6x + 12x - 18x = 3 - 9 + 12 \to 0 = 6.$$

Thus we have no solution. For $k = -3$we get

$$\begin{aligned} 6x - 2y - 9z &= 3 \\ 6x + y &= -6 \\ -6x + 3z &= 3. \end{aligned}$$

Solving for z we have $z = 1 + 2x$ and solving for y we have $y = -6 - 6x$ and putting into the 1st we have

$$6x + 12x - 18x = 3 + 9 - 12 \to 0 = 0.$$

This is true, thus there are infinitely many solutions.

1.6 Let $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} i \\ 0 \\ -i \end{pmatrix}$ and define $A = vw^*$, where $*$ denotes the complex conjugate of the transpose. (a) Compute A, (b) What is the rank of A? Give a basis for its range, (c) Find a SVD of A, (d) Find the pseudo-inverse of A.

We have

$$A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -i & 0 & i \end{pmatrix} = \begin{pmatrix} -i & 0 & i \\ -i & 0 & i \\ -i & 0 & i \end{pmatrix}.$$

A is clearly rank 1 as we can reduce it to a single column quite easily. What is a basis for the range of A? Well, we want vectors so that linear combinations of them can produce anything in the range of A. We have

$$Ax = \begin{pmatrix} -ix_1 + ix_3 \\ -ix_1 + ix_3 \\ -ix_1 + ix_3 \end{pmatrix}$$

thus the basis for anything in the range of A is simply a scalar vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Now, for (c), to find the SVD we will go straight from the definition of the SVD. First form

$$A^*A = \begin{pmatrix} i & i & i \\ 0 & 0 & 0 \\ -i & -i & -i \end{pmatrix} \begin{pmatrix} -i & 0 & i \\ -i & 0 & i \\ -i & 0 & i \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 3 \end{pmatrix}$$

and find the eigenvalue decomposition. This isn't terribly hard and we have:

$$\begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = 6 \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

Now, remember for SVD we use ONLY the nonzero eigenvalues of A^*A. Thus $V = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ and $\Sigma = \sqrt{6}$. To find U, we have $U_i = AV_i/\sigma_i$ and since we have only one entry, this is easy, giving us

$$U = \begin{pmatrix} -i & 0 & i \\ -i & 0 & i \\ -i & 0 & i \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} / \sqrt{6} = \begin{pmatrix} i/\sqrt{3} \\ i/\sqrt{3} \\ i/\sqrt{3} \end{pmatrix}$$

and

$$A = U\Sigma V^* \begin{pmatrix} i/\sqrt{3} \\ i/\sqrt{3} \\ i/\sqrt{3} \end{pmatrix} \sqrt{6} \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}^T$$

As for (d), to compute the pseudo-inverse given an SVD, we simply do

$$A^+ = V\Sigma^{-1}U^*$$

where Σ^+ is contains the inverse of the entries in Σ. Thus we have

$$A^+ = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} -i/\sqrt{3} \\ -i/\sqrt{3} \\ -i/\sqrt{3} \end{pmatrix}^T.$$

1.9 Consider the differential equation $y' = \lambda y$, where $\lambda \in \mathbb{C}$, approximated by the finite difference scheme,

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left(y'_n + y'_{n+1}\right) + \frac{\Delta t^2}{12}\left(y''_n - y''_{n+1}\right).$$

(a) Determine the order of accuracy of this scheme; (b) Show that the region of absolute stability contains the entire negative real axis, $\lambda < 0$.

This problem seems a bit odd, since rarely would you see a scheme like this with iteration on the derivatives as well. Let us proceed with some standard Taylor series formulas to find the accuracy. We have

$$\begin{aligned} y_{n+1} &= y_n + \Delta t y'_n + \frac{\Delta t^2}{2} y''_n + \frac{\Delta t^3}{6} y'''_n + \frac{\Delta t^4}{24} y^{(4)}_n + O\left(\Delta t^5\right) \\ y'_{n+1} &= y'_n + \Delta t y''_n + \frac{\Delta t^2}{2} y'''_n + \frac{\Delta t^3}{6} y^{(4)}_n + O\left(\Delta t^4\right) \\ y''_{n+1} &= y''_n + \Delta t y'''_n + \frac{\Delta t^2}{2} y^{(4)}_n + O\left(\Delta t^3\right). \end{aligned}$$

Note that we only go to 4^{th} order on y' and 3^{rd} order on y'' since they are multiplied by Δt and Δt^2, respectively. Now lets toss everything in there and collect terms

$$\begin{aligned} y_{n+1} - y_n &= \Delta t y'_n + \frac{\Delta t^2}{2} y''_n + \frac{\Delta t^3}{6} y'''_n + \frac{\Delta t^4}{24} y^{(4)}_n + O\left(\Delta t^5\right) \\ -\frac{\Delta t}{2}\left(y'_n + y'_{n+1}\right) &= -\Delta t y'_n - \frac{\Delta t^2}{2} y''_n - \frac{\Delta t^3}{4} y'''_n - \frac{\Delta t^4}{12} y^{(4)}_n + O\left(\Delta t^5\right) \\ -\frac{\Delta t^2}{12}\left(y''_n - y''_{n+1}\right) &= \frac{\Delta t^3}{12} y'''_n + \frac{\Delta t^4}{24} y^{(4)}_n + O\left(\Delta t^5\right). \end{aligned}$$

Adding each of these terms together gives us our method (we brought everything to the LHS). We clearly get the Δt and Δt^2 terms canceling in the first two equations. We are left with

$$\frac{\Delta t^3}{6}y_n''' - \frac{\Delta t^3}{4}y_n''' + \frac{\Delta t^3}{12}y_n''' + \frac{\Delta t^4}{24}y_n^{(4)} - \frac{\Delta t^4}{12}y_n^{(4)} + \frac{\Delta t^4}{24}y_n^{(4)} + O\left(\Delta t^5\right)$$

The first three terms cancel, and clearly the Δt^4 terms do not, thus we are left with $O\left(\Delta t^4\right)$.

As for A-stability, I believe we simply take $y' = \lambda y$ and $y'' = \lambda y' = \lambda^2 y$ and plug them into our scheme. We get something like

$$y_{n+1} - \frac{\Delta t}{2}\lambda y_{n+1} + \frac{\Delta t^2}{12}\lambda^2 y_{n+1} = y_n + \frac{\Delta t}{2}\lambda y_n + \frac{\Delta t^2}{12}\lambda^2 y_n.$$

Letting $z = \Delta t \cdot \lambda$ this can be written as

$$y_{n+1}\left(1 - \frac{z}{2} + \frac{z^2}{12}\right) = y_n\left(1 + \frac{z}{2} + \frac{z^2}{12}\right)$$

or

$$y_{n+1} = y_n \frac{\left(1 + \frac{z}{2} + \frac{z^2}{12}\right)}{\left(1 - \frac{z}{2} + \frac{z^2}{12}\right)}.$$

We have A-stability iff

$$\left|\frac{1 + \frac{z}{2} + \frac{z^2}{12}}{1 - \frac{z}{2} + \frac{z^2}{12}}\right| < 1$$

for $Re(\lambda) < 0$. Hopefully the complex analysis you've learned in the other qualifier courses helps you give a good argument to why this is true at this point.

1.11 For each of the following IVP problems below, indicate which of the following methods would be the most suitable choice for use in numerical integration. In each case, indicate why your choice is best, and what it is about the other methods that makes them less desirable. Assume that in all cases you are interested in finding the value of the solution at $t = 10$, with an absolute error no worse than $\approx 10^{-4}$. If the method you select has a fixed step size, indicate roughly what step size you believe will be necessary to produce the desired accuracy.

Methods: Fourth order Adaptive RK, BDF4 (Fixed step size fourth order based on a backward difference), Fixed step size Euler, AB4/AM4: Predictor corrector (with fixed step).

Initial Value Problems:

(a) $y' = 106\left(\cos(t) - y\right)$ with $y(0) = 1$.

(b) $y' = (1 - y) + 100e^{-(100(t-5))^2}$ with $y(0) = 0$.

(c) $y' = -\tanh\left(\frac{\lfloor 100y \rfloor}{100}\right)$ with $y(0) = 1$.

I'm going to go ahead and assume Fixed step size Euler means forward Euler. Recall we can have fwd or bkwd Euler. Backward would be implicit, forward is explicit. Let us quickly recall the properties of each method.

Fourth order Adaptive RK would be an explicit method that alternates between a 4^{th} and 5^{th} order method, using the error in each to compute new step sizes. This would be useful for non-stiff problems where we would have some small time period where we would would have to account for some rough effects. In other words, it would be great for a problem that would work with a fixed step size, except for some certain domains. In this case, using a fixed step size, we'd have to take the step size to be the smallest needed for the entire domain, slowing down the computations for the nice smooth areas. Recall that no explicit RK method is A-stable.

BDF4 would be very useful for stiff problems since it would be more stable than the AB4/AM4 methods. The problem with using BDF4 instead of AB4/AM4 is that we must solve an implicit system of course. I would suggest using AB4/AM4 in problems that are not terribly stiff. In a real world application you'd derive the regions of absolute stability for AB4/AM4 (they are never fully A-stable, and for 4^{th} order like this, they will have a very small region in complex space where they are A-stable). As for forward Euler the only advantage is it's cost for a problem where you can use a nice large step size and there are no problems with stiffness.

I'm just going to go through this quickly, since most of the discussion is above. For (a) we've got

$$f(t, y) = 106\left(\cos(t) - y\right).$$

This is going to be relatively smooth, but will have some large values. The problem isn't exactly going to get smoother as time runs through, so I'd say RK wouldn't be much help. There doesn't seem to be a need to use BDF so I'd go with Euler of AB4/AM4. Euler may work, but it is of course only 1^{st} order, so you'd have to take a small step size for 10^{-4} error. AB4/AM4 are 4^{th} order, so not nearly as small a step size. I highly doubt they would expect you to know the exact truncation errors for each of these methods so I'd just use the order to estimate step size.

For (b) we end up with a bit of a problem, since $100e^{-(100(t-5))^2}$ is ridiculously close to zero except right around $t = 5$, when it's right around 100. This is going to create serious problem, so we'd have to go with something that can handle this type of fluctuation, which I'd guess would be BDF4. Again, use the fact it's 4^{th}order to estimate the step size.

For (c) just recall what tanhlooks like. It has a steep derivative around 0 and then it levels off as the values get larger. So this is going to be a 'rough' right hand side for a while around 0, then it'll get nice and level off. Even though there is this floor function, it should not matter very much once we get away from 0. For this reason, I'd use the RK since you can take advantage of the large step sizes away from 0. As for the error, since the step size changes, you may just want to do a general idea for 4^{th} order like in the other ones.

2. For some reason, I didn't find any numerical analysis questions for day 2. I had loose sheets, so perhaps they were mislabeled as another exam and I've done them as part of that.

August 2006

1.5 Find the solution of $\dot{x}(t) = Ax(t),\ x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ where $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}$.

There are a couple of ways to do this. If A is diagonalizable, you can find P so that $P^{-1}AP = \Lambda$ where Λ is diagonal, change the DE to $Y' = \Lambda Y$ (here $x = PY$) solve this, then write $x = PY = Pe^{\Lambda t}P^{-1}x(0)$. Another way is to go straight from the matrix exponential definition, where $x(t) = e^{At}x(0)$. The problem here, of course, is that

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \dots$$

and I'm doubting they'll want your solution in terms of a series.

I'd go with the first approach, but the problem is that this matrix is not diagonalizable. If I remember correctly, we rewrite A as

$$A = 2I + H$$

where H contains the upper triangular part. Then

$$e^{At} = e^{2t}e^{H} = e^{2t}\left(I + Ht + \frac{1}{2!}H^2t^2 + ...\right).$$

Now,

$$H = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix},\ H^2 = \begin{pmatrix} 0 & 0 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ H^3 = 0 \cdot I.$$

So

$$e^{At} = e^{2t}\begin{pmatrix} 1 & 3t & 4t + \frac{15}{2}t^2 \\ 0 & 1 & 5t \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$x(t) = e^{At}x(0) = e^{2t}\begin{pmatrix} 1 + 7t + \frac{15}{2}t^2 \\ 1 + 5t \\ 1 \end{pmatrix}.$$

Let us test this out:

$$x'(t) = 2e^{2t}\begin{pmatrix} 1 + 7t + \frac{15}{2}t^2 \\ 1 + 5t \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 7 + 15t \\ 5 \\ 0 \end{pmatrix} = e^{2t}\begin{pmatrix} 9 + 29t + 15t^2 \\ 7 + 10t \\ 2 \end{pmatrix}$$

and

$$Ax(t) = e^{2t}\begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 + 7t + \frac{15}{2}t^2 \\ 1 + 5t \\ 1 \end{pmatrix} = e^{2t}\begin{pmatrix} 2 + 14t + 15t^2 + 3 + 15t + 4 \\ 2 + 10t + 5 \\ 2 \end{pmatrix} = e^{2t}\begin{pmatrix} 9 + 29t + 15t^2 \\ 7 + 10t \\ 2 \end{pmatrix}$$

1.9 P is a projection matrix if $P^2 = P$. If P is an $n \times n$ projection matrix, show that P is diagonalizable. *Hint:* First show $\mathbb{R}^n = \ker(P) \oplus \mathrm{Im}(P)$, then use this result to show there is a basis of eigenvectors.

There seems like there should be a nice trick to make this a one or two liner, but I can't see it right away. I believe the first step is to find the eigenvalues of P. Well, $Px = \lambda x = P^2 x = \lambda P x = \lambda^2 x$ so the only eigenvalues are 0 and 1. Now, the hint is the same thing as showing an vector $v \in \mathbb{R}^n$ can be written as $v = w + x$ with $w \in \mathrm{Im}(P)$ (or the span of the eigenspace corresponding to 1) and $x \in \ker(P)$ (the span of the eigenspace corresponding to 0). Let $w = Pv$ and $x = v - Pv$. Since the total space of the eigenvectors must be n and we can represent any $v \in \mathbb{R}^n$, this shows we have a linearly independent set of eigenvectors.

1.10 Show that the eigenvectors of a real symmetric matrix are the stationary points of the Rayleigh quotient.

First, I have no clue what it means by the stationary points of the Rayleigh quotient. I would assume this means that we want to show

$$R(x) = \frac{(Ax, x)}{(x, x)} = \lambda$$

for any eigenvectors x, although I'm not sure what there is to actually show. If we let λ_i, x_i be an eigenpair of real symmetric A then

$$\frac{(Ax_i, x_i)}{(x_i, x_i)} = \frac{(\lambda_i x_i, x_i)}{(x_i, x_i)} = \lambda_i.$$

I guess the hard part of the problem is showing that these are the only stationary points. Well, what does a stationary point mean? I would assume this means $\nabla R = 0$.

$$\nabla R = \frac{2Ax - 2R(x)x}{(x, x)} = 0$$

which implies $Ax = R(x) \cdot x$ or that $R(x)$ is an eigenvalue of A.

1.11 Describe a general algorithm and write a pseudo-code for the reduction of a real symmetric matrix to the Hessenberg form by means of orthogonal similarity transformations. Prove the resulting matrix is tridiagonal.

This is a Schur factorization. Use Householder. So you multiply by

$$Q_k = \begin{pmatrix} I_{k-1 \times k-1} & 0 \\ 0 & F \end{pmatrix}$$

where the F is selected to introduce zeros into the k^{th}column of the matrix A. Recall the idea here is to choose F to be a Householder reflector. Simply take a look at the standard Householder QR in any linear algebra book. Since we are reducing to Hessenberg form, we wish to do this in a symmetric manner. So we take A and perform

$$Q_1^* A Q_1$$

in order from left to right. We continue this process until a Hessenberg matrix is formed (until $n-2$). Why is this tridiagonal? It's quite easy, we have $Q^* A Q = H$ where H is the Hessenberg. Recall that since it's Hessenberg, that means it is almost upper triangular, with nonzero entries only on the subdiagonal and above. Since A is symmetric we have

$$(Q^* A Q)^* = Q^* A Q = H = H^*.$$

Since $H = H^*$ this means we must have a tridiagonal matrix, since H^* Will have nonzero entries only on the superdiagonal and below, while H has nonzeros only on the subdiagonal and above.

2.3 Suppose that the square system $Ax = b$ is solved by some backward stable algorithm so that the computed solution $\tilde{x}$ satisfies $\tilde{A}\tilde{x} = b$ with square matrix $\tilde{A} = A + \delta A$ such that $\|\delta A\| / \|A\| = O(\varepsilon_{mach})$. Find an estimate on the error in $\tilde{x}$ if the condition number $\kappa(A)$ of the matrix is known.

We want to estimate $\|x - \tilde{x}\|$:

$$\begin{aligned} \|x - \tilde{x}\| &= \left\| A^{-1}b - \tilde{A}^{-1}b \right\| = \left\| A^{-1}b \left(1 - \frac{1}{1+\delta}\right) \right\| = \left\| A^{-1}b \left(\frac{\delta}{1+\delta}\right) \right\| \leq \left\| A^{-1} \left(\frac{\delta A}{1+\delta}\right) x \right\| \\ &\leq \left\| A^{-1} \right\| \|\delta A\| \|x\| = \left\| A^{-1} \right\| \|A\| \frac{\|\delta A\|}{\|A\|} \|x\| = C \cdot \kappa(A) \|x\| \varepsilon_{mach} \end{aligned}$$

Well, this isn't very good since we explicitly have $\|x\|$ on the right hand side. I guess we can use $\|A\|\|x\| \leq \|b\|$ and write $\|x\| \leq \|b\|\|A\|^{-1}$ also, but this isn't terribly great either. At least this may give you some idea of how to get a nice estimate.

2.4 For an arbitrary square matrix $A \in \mathbb{C}^{m \times m}$ and norm $\|\cdot\|$, prove that $\lim_{n \to \infty} \|A^n\| = 0$iff $\rho(A) < 1$ where $\rho$is the spectral radius. *Hint:* Use Schur decomposition or the Jordan normal form.

Lets use the hint. Schur decomposition says that A is unitarily similar to a triangular matrix. Thus we can write

$$A = UTU^*$$

where T is upper triangular and has the eigenvalues of A on the diagonal. Since U is unitary, we have

$$A^n = UT^nU^*$$

since T^n has the eigenvalues on the diagonal, this diagonal will blow up if $\rho(A) > 1$. That's the idea here.

2.6 The following problem arises in the study of linear regression in statistics. Let M be an n by k real matrix with $n \geq k$. We assume that the null space of M contains only the zero vector.

(a) Prove that M^TM is invertible. (b) Given $y \in \mathbb{R}^n$, we want to find the vector $x \in \mathbb{R}^k$ which minimizes $\|y - Mx\|$. Show that the minimizer is given by $x = \left(M^TM\right)^{-1} M^Ty$.

For (a), since the null space of M contains only the zero vector, it means $x = 0$ is the only vector such that $Mx = 0$. Hence $M^TMx = 0$ for $x = 0$ as well. If $M^TMx = 0$ then $x^TM^TMx = 0$ which implies $\|Mx\| = 0$ which is true only if x is in the null space of M, hence $x = 0$. Thus M^TM is invertible.

For (b), we want to minimize $G(x) = \|y - Mx\|^2 = (y - Mx)^T (y - Mx)$. Taking the derivative and setting equal to zero, we get

$$2M^TMx - 2M^Ty = 0.$$

Solving for x we have

$$x = \left(M^TM\right)^{-1} M^Ty.$$

For (b), I might use the inner-products as well to make a quadratic form: $x^TKx - 2f^Tx + c$ which must have a minimum at $x = K^{-1}f$. I assume they want the Euclidean norm here. The same type of proof should work for any induced norm.

2.7 (1) Describe (shifted) inverse iteration algorithm for finding the eigenvalues of a matrix A which are close to a given value μ. (2) Show that, if λ_j is the closest eigenvalue to the shift μ, and λ_k is the next closest eigenvalue ($|\mu - \lambda_j| < |\mu - \lambda_k| \leq |\mu - \lambda_i|$, $i \neq j, k$), then the iterates satisfy

$$\left|\lambda^{(m)} - \lambda_j\right| = O\left(\left|\frac{\mu - \lambda_j}{\mu - \lambda_k}\right|^{2m}\right),$$

as $k \to \infty$.

This is not something common with Juan's questions, so I'm gonna go ahead and skip this for now. If you wanted to go ahead and do this, just look at the power algorithm for finding eigenvalues.

2.8 (1) Find all methods of the form

$$\sum_{k=0}^{2} a_k y_{n-k} = \Delta t \sum_{k=0}^{2} b_k f\left(t_{n-k}, y_{n-k}\right)$$

where $t_m = m\Delta t$ for solving $y' = f(t, y)$ with initial value $y(0) = y_0$ which are at least order 2 (give a parametrization). Assume that $a_0 = 1$. (2) Which of these methods are consistent? (3) Which are explicit?

To find the order of the multistep method, we use the standard Taylor series type technique.

$$0: \sum_{k=0}^{2} a_i = 0 \to 1 + a_1 + a_2 = 0.$$

Since they decided to be jerks on this problem and make a_0 correspond to the highest term, we need to change the usual definition a little. We normally have

$$1: \sum_{i=0}^{2} (ia_i - b_i) = 0$$

which we need to rearrange a bit to write

$$1: \sum_{i=0}^{2} \left((2 - i)\, a_i - b_i\right) = 2 + a_1 - b_0 - b_1 - b_2 = 0.$$

Similarly

$$2: \sum_{i=0}^{2} \left(\frac{1}{2} (2-i)^2 a_i - (2-i) b_i \right) = 1 + \frac{a_1}{2} - 2b_0 - b_1 = 0.$$

It seems to me you've got three equations and 5 unknowns, so I'd think you'd have two free parameters.

For consistency, recall that you can associate two polynomials to the multistep method:

$$p(z) = 1 \cdot z^2 + a_1 z + a_2, \; q(z) = b_0 z^2 + b_1 z + b_2.$$

To be consistent, we need $p(1) = 0$ and $p'(1) = q(1)$. This is

$$\begin{aligned} p(1) &= 0 = 1 + a_1 + a_2 \\ p'(1) = q(1) &= 2 + a_1 = b_0 + b_1 + b_2. \end{aligned}$$

Now, this is the same as having 0^{th} and 1^{st} order satisfied in the Taylor series. As for the explicit method, what does this mean? It means the left hand side is one order higher than the right. So any explicit method would need to have $b_0 = 0$.

January 2006

1.4 Let $v, w \in \mathbb{C}^m$ and define

$$A = vw^* \in \mathbb{C}^{m \times m}.$$

(a) Find all eigenvalues and eigenvectors of A. (b) Find the singular values and left and right singular vectors for A.

Hopefully you know by now that the product of a column vector and a row vector is a rank 1 matrix, so we're only looking for one eigenvalue. Eigenpairs satisfy

$$Ax = \lambda x = vw^* x$$

or

$$x^* A = \lambda x^* = x^* v w^*.$$

Now, the thing here is that $w^* x = (w, x)$ which is a scalar. Similarly $x^* v = (x, v)$. There is a bit of an issue here, since often eigenvectors really means only right eigenvectors, but to be safe you may want to consider both. Let us just look at the right. We have

$$Ax = \lambda x = v\,(w, x)$$

thus the eigenvector has to be a scalar multiple of v. So we have

$$Av = vw * v = (w, v)v.$$

The eigenvalue for the left eigenvector is the same, and clearly the left eigenvector is w.

What are singular values/vectors? They are the eigenvectors and square root of the eigenvalues of A^*A.

$$A^*A = wv^* vw^* = (v, v) ww^*.$$

This is, again, rank 1. Clearly the eigenvectors are w with eigenvalue $(v, v)(w, w)$.

1.5 Let $B = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix}$. (a) Find the PLU factorization of B. ($PB = LU$). (b) Find a QR factorization of B ($B = QR$).

PLU is an LU factorization with pivoting P. Recall an LU factorization is a factorization into a lower and upper triangular matrix. We can make a bit of a choice with any LU factorization in terms of the diagonal. We can let either the L or the U have 1 on the diagonal (note for Hermitian matrices we can take the sqrt of the eigenvalues on the diagonal). Now, to do this for any general problem, simply take B and row reduce it while keeping track of pivots. The resulting U is the upper triangular part, then assuming the diagonal of L is 1, we fill in the L values by enforcing $PLU = A$. This one we can just do directly, by assuming

$$L = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix}, U = \begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix}$$

then

$$LU = \begin{pmatrix} u_1 & u_2 \\ lu_1 & lu_2 + u_3 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \text{ or } \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$$

where the 2nd matrix would be with pivoting $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Lets try to solve for the first (no pivoting). We get

$$u_1 = 2,\ u_2 = -3,\ l = 4/2 = 2,\ u_3 = -5 + 6 = 1.$$

You can do something similar for the pivot. I'm not sure if they NEED there to be pivoting but I'd go ahead and do the version with $P \neq I$.

Now, QR factorization should be pretty automatic at this point. Q is going to be a unitary matrix and R is going to be uppertriangular. I would guess there is a nice trick to do this for just a 2×2, but we'll go ahead and use Gram-Schmidt. We take

$$u_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix},\ e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Then

$$u_2 = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \frac{e_1^T \begin{pmatrix} -3 \\ -5 \end{pmatrix}}{e_1^T e_1} e_1 = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + \frac{13}{\sqrt{5}} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + \frac{13}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ e_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Thus

$$Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Then

$$R = Q^T B = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 10 & -13 \\ 0 & 1 \end{pmatrix}.$$

1.6 Add comments to the Matlab code below... etc.

I'm not gonna bother with this. It looks a whole lot like a Gram-Schmidt algorithm. Take a look at that if you're stuck.

1.12 Find a polynomial that interpolates the values of a function $f(x)$ at points $0, h/4$ and $3h/4$ and use it to estimate the value of $\int_0^h f(x)\,dx$. What order error does this estimate have?

So, we're gonna use Lagrange polynomials:

$$\begin{aligned} P(x) &= f(0) \frac{x - 3h/4}{0 - 3h/4} \frac{x - h/4}{0 - h/4} + f(h/4) \frac{x - 0}{h/4 - 0} \frac{x - 3h/4}{h/4 - 3h/4} + f(3h/4) \frac{x - 0}{3h/4 - 0} \frac{x - h/4}{3h/4 - h/4} \\ &= f(0) \frac{16}{3h^2} \left(x - \frac{3h}{4}\right) \left(x - \frac{h}{4}\right) - f(h/4) \frac{16}{2h^2} x \left(x - \frac{3h}{4}\right) + f(3h/4) \frac{16}{6h^2} x \left(x - \frac{h}{4}\right). \end{aligned}$$

For the integral:

$$\int_0^h f(x)\,dx \approx \int_0^h P(x)\,dx = \frac{h}{9} f(0) + \frac{h}{3} f\left(\frac{h}{4}\right) + \frac{5h}{9} f\left(\frac{3h}{4}\right).$$

As for the error estimate, this is based on the standard interpolation error formula

$$f(x) - p(x) = \frac{f^{(3)}(\xi)}{3!} x \left(x - \frac{h}{4}\right) \left(x - \frac{3h}{4}\right).$$

We assume $\exists\ M$ st $\left|f^{(3)}(\xi)\right| \leq M$ and integrate. We get

$$\left| \int_0^h f(x) - p(x)\,dx \right| \leq \frac{M}{3!} \int_0^h \left| x \left(x - \frac{h}{4}\right) \left(x - \frac{3h}{4}\right) \right| dx = \frac{Mh^4}{192}.$$

2.6 Find values of c_0, c_1 and c_2 so that

$$u_n - u_{n-2} = 2\Delta t \left(c_0 f(t_n, u_n) + c_1 f(t_{n-1}, u_{n-1}) + c_2 f(t_{n-2}, u_{n-2})\right)$$

is 3^{rd} order accurate in integrating $u' = f(t, u)$. Describe the stability properties of the method you find.

This is yet another multistep problem. We're gonna recast this in the form

$$\sum_{k=0}^{2} a_{2-k} u_{n-k} = \Delta t \sum_{k=0}^{2} b_{2-k} f_{n-k}$$

have $a_0 = -1$, $a_1 = 0$, $a_2 = 1$, $b_0 = 2c_2$, $b_1 = 2c_1$ and $b_2 = 2c_0$. Then for 3^{rd} order accurate, we need

$$0: \sum_{i=0}^{2} a_i = 0$$

$$1: \sum_{i=0}^{2} (ia_i - b_i) = 1 - b_0 - b_1 - b_2 = 0$$

$$2: \sum_{i=0}^{2} \left(\frac{1}{2!} i^2 a_i - i b_i \right) = 2 - b_1 - 2b_2 = 0$$

$$3: \sum_{i=0}^{2} \left(\frac{1}{3!} i^3 a_i - \frac{1}{2!} i^2 b_i \right) = \frac{4}{3} - \frac{b_1}{2} - 4b_2 = 0.$$

You've got 3 equations and 3 unknowns so just solve for the b values and then get c values from this. As for stability, we right the left size in a polynomial as $p(z) = z^2 - 1$ and we need to show that the roots lie in $|z| \leq 1$ and roots with $z = 1$ are simple. Clearly $z^2 = 1$ at 1 and -1 and both are simple roots.

August 2005

1.5 Let A be a real $m \times m$ matrix and let $B = A^T A$. Consider the sequence

$$x_n = (\text{Trace} B^n)^{1/n}$$

and show that

$$\lim_{n \to \infty} x_n = \|A\|^2$$

where $\|A\|$ is the standard matrix norm.

First, I have no idea what they mean by the standard matrix norm (there is no standard matrix norm to me...), but we'll try to show this sequence converges to some limit and hopefully we'll see what norm comes our from that. There are two big things to notice here: 1.B is symmetric so we can use the Schur factorization to write $B = U \Lambda U^*$ and 2. The trace operator is invariant under similarity transformations. So $B^n = U\ \Lambda^n U^*$and

$$tr\,(B^n) = tr\,(U\ \Lambda^n U^*) = tr\,(\Lambda^n U\ U^*) = tr\,(\Lambda^n) = \sum \lambda_i^n.$$

Now we want to show

$$\lim_{n \to \infty} \left(\sum \lambda_i^n \right)^{1/n} = \|A\|^2.$$

I guess I'd use some kinda of spectral radius argument here and conclude that the left hand side converges to $\lambda_{max} = \rho(B) = \rho(A^T A)$. Then the right side would be the square of the spectral norm.

1.7 (a) Show that $\phi = \dfrac{1 + \sqrt{5}}{2}$ is a stable fixed point of the map $x \to \dfrac{x+1}{x}$. (b) What is the rate of convergence?. (c) Give an example of a nontrivial iteration which converges quadratically to ϕ.

This looks like more applied analysis, but I'm not quite sure. Regardless I'm not going to spend a lot of time on this. For (a) I would think you use a fixed point theorem, call $f(x) = \dfrac{x+1}{x}$ and show $|f'(x)| < 1$ in a neighborhood of ϕ. For (b) I would think you can again use an argument based on the bounds of the derivative. For (c) you may want to try writing as $g(x) = x - \dfrac{x+1}{x}$ and suggesting Newton's method.

1.8 Add comments to the Matlab ... etc

I don't really like to spend a whole lotta time on these since they're quite easy. The key here should be that you are starting with a mystery function returning matrices L, U and P. I would assume to start that this is an LU decomposition. Which one? Well, L has ones on the diagonal, so the Doolittle algorithm (with pivoting... because of the P).

1.11 Find the projection of the vector $u = \begin{bmatrix} 2 & 2 & 2 & 0 \end{bmatrix}^T$ on the subspace spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Let $V = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. We want the projection of u onto the space spanned by the columns of V so we solve

$$V^T V x = V^T u.$$

This will define orthogonal projection, which I assume is what they want. We have

$$x = \left(V^T V\right)^{-1} V^T u = \begin{bmatrix} 3/2 \\ 1 \\ 1/2 \end{bmatrix}.$$

This is not a projection, we want to project x onto the range of V so we get

$$Vx = \begin{bmatrix} 3/2 \\ 5/2 \\ 3/2 \\ 1/2 \end{bmatrix}.$$

We know this is a projection because our operator that takes u into the answer above is

$$P = V\left(V^T V\right)^{-1} V^T$$

and note

$$P^2 = V\left(V^T V\right)^{-1} V^T V\left(V^T V\right)^{-1} V^T = V\left(V^T V\right)^{-1} I V^T = P$$

so we have a projection.

1.12 (a) Write the Lagrange polynomial that interpolates a function $f(x)$ whose values are known at point $x = 0$, Δ and 3Δ for some $\Delta > 0$. (b) Use your answer from part (a) to estimate the value of $f'(0)$ in terms of the known values of f with an error term in the big O notation. Explain how you know the error's order in terms of Δ, an give an example to show that your estimate is sharp.

You should be good at Lagrange polynomials by now:

$$\begin{aligned} P(x) &= f(0)\frac{x-3\Delta}{0-3\Delta}\frac{x-\Delta}{0-\Delta} + f(\Delta)\frac{x-0}{\Delta-0}\frac{x-3\Delta}{\Delta-3\Delta} + f(3\Delta)\frac{x-0}{3\Delta-0}\frac{x-\Delta}{3\Delta-\Delta} \\ &= f(0)\frac{1}{3\Delta^2}(x-3\Delta)(x-\Delta) - f(\Delta)\frac{1}{2\Delta^2}x(x-3\Delta) + f(3\Delta)\frac{1}{6\Delta^2}x(x-\Delta). \end{aligned}$$

How do we estimate $f'(0)$? Well $f(x) \approx P(x)$ so we'll assume $f'(0) \approx P'(0)$.

$$\begin{aligned} f'(0) &\approx P'(0) = f(0)\frac{1}{3\Delta^2}[(0-3\Delta)+(0-\Delta)] - f(\Delta)\frac{1}{2\Delta^2}[0+(0-3\Delta)] + f(3\Delta)\frac{1}{6\Delta^2}[0+(0-\Delta)] \\ &\approx f(0)\frac{1}{3\Delta^2}[-4\Delta] - f(\Delta)\frac{1}{2\Delta^2}[-3\Delta] + f(3\Delta)\frac{1}{6\Delta^2}[-\Delta] = \frac{1}{\Delta}\left(\frac{-4}{3}f(0) + \frac{3}{2}f(\Delta) - \frac{1}{6}f(3\Delta)\right). \end{aligned}$$

As for error estimates, start with the usual interpolation error

$$f(x) - p(x) = \frac{f^{(3)}(\xi)}{3!}x(x-\Delta)(x-3\Delta)$$

take a derivative (ξ is dependent on x) and plug in $x = 0$

$$f'(0) - p'(0) = \frac{f^{(4)}(\xi)}{3!}[(0-\Delta)(0-3\Delta) + 0(0-\Delta) + 0(0-3\Delta)] = \frac{f^{(4)}(\xi)}{3!}3\Delta^2 = \frac{f^{(4)}(\xi)\Delta^2}{2}.$$

So

$$|f'(0) - p'(0)| = O\left(\Delta^2\right).$$

I'm not exactly sure what they want for an example to show this is sharp. You can take something which has a vanishing 3^{rd} derivative and show this is exact, or you can take something silly like $f(x) = \sin(4x)$, $\Delta = \pi/4$ so that $p(x) = 0$. To me, the easiest thing to do is just to interpolate another polynomial. Let us say $f(x) = x^4$ for $x = 0, 1, 3$, then $p(x) = x(13x - 12)$. So $f'(0) = 0$ and $p'(0) = -12$. Then

$$|f'(0) - p'(0)| = 12 \leq \frac{f^{(4)}(\xi)\Delta^2}{2} = \frac{12}{2}\cdot 1^2$$

so we have a sharp estimate.

2.4 A multi-step linear method for approximately solving ODEs is given by

$$y_n = y_{n-2} + h\left(\frac{7}{3}f_{n-1} - \frac{2}{3}f_{n-2} + \frac{1}{3}f_{n-3}\right).$$

(a) Is this method consistent? Of what order? (Show your reasoning)

(b) Is this method 0-stable? Strongly stable? Explain.

Since one of these pops up on pretty much every exam, you oughtta be good at these. (a) is asking for the order, so we rewrite as

$$y_n - y_{n-2} = h\left(\frac{7}{3}f_{n-1} - \frac{2}{3}f_{n-2} + \frac{1}{3}f_{n-3}\right)$$

and I define (from lowest to highest terms) $a_0 = 0$, $a_1 = -1$, $a_2 = 0$ and $a_3 = 1$. Then $b_0 = \frac{1}{3}$, $b_1 = \frac{-2}{3}$, $b_2 = \frac{7}{3}$ and $b_3 = 0$. Then go through the orders as usual:

$$\begin{aligned}
0: &\sum_{i=0}^{3} a_i = 0\\
1: &\sum_{i=0}^{3} (ia_i - b_i) = -1 + 3 - \frac{1}{3} + \frac{2}{3} - \frac{7}{3} = 0\\
2: &\sum_{i=0}^{3} \left(\frac{1}{2!}i^2a_i - ib_i\right) = -\frac{1}{2} + \frac{9}{2} + \frac{2}{3} - \frac{14}{3} = 0\\
3: &\sum_{i=0}^{3} \left(\frac{1}{3!}i^3a_i - \frac{1}{2!}i^2b_i\right) = -\frac{1}{6} + \frac{9}{2} + \frac{1}{3} - \frac{14}{3} = 0\\
4: &\sum_{i=0}^{3} \left(\frac{1}{4!}i^4a_i - \frac{1}{3!}i^3b_i\right) = -\frac{1}{24} + \frac{27}{8} + \frac{1}{9} - \frac{28}{9} = \frac{1}{3}
\end{aligned}$$

It looks to be a 3^{rd} order method. For stability, I believe by 0−stable they mean regular stability. So we write the left hand side in polynomial form as

$$p(z) = z^3 - z = z(z-1)(z+1)$$

and find the zeros are either so that $|z| < 1$ or are simple, thus this is stable. Strongly stable means the only root with modulous 1 is $z = 1$. This is not true (because of $z = -1$) thus this is not strongly stable.

2.5 Set up the stiffness matrix A and load vector b resulting from discretizing the BVP

$$-u'' + 2u = 1,\ t \in (0,1),\ u'(0) = u'(1) = 0$$

using hat functions on a mesh $\{0, 1/2, 1\}$. Your answer should be in the form $Ax = b$ with explicit numerical values given for the entries of A and b. Explain how your x values produce and approximation to u. (NOTE: This is redone, it was not correct the first time around)

First, you need a variational formulation. We multiply by v and integrate over $(0,1)$ to get

$$\int_0^1 -u'' \cdot v\,dx + 2\int_0^1 uv\,dx = \int_0^1 v.$$

Integrate by parts and use the fact that $u'(0) = u'(1) = 0$ to get

$$\int_0^1 u'v'\,dx + 2\int_0^1 uv\,dx = \int_0^1 v.$$

This first part, $\int_0^1 u'v'\,dx$, is what will form our stiffness matrix upon discretization. The second part would form our mass matrix. The concept of solving $Ax = b$ like the problem states doesn't make much sense here since we really do have a mass matrix, so perhaps theyjust want us to derive on the stiffness matrix.

With a finite element method, normally we would define a map from a reference element to the grid and whatnot, but in 1D with four piecewise linears, this is really not necessary. We're going to have 4 linear basis functions:

$$\begin{aligned}
&\text{on } (0,1/2): \qquad \psi_1 = -2x + 1,\ \psi_2 = 2x\\
&\text{on } (1/2,1): \qquad \psi_3 = -2x + 2,\ \psi_4 = 2x - 1.
\end{aligned}$$

These will turn into 3 basis functions (hat functions):

$$\begin{aligned}\phi_1 &= -2x+1,\ x\in(0,1/2)\\ \phi_2 &= \left\{\begin{array}{c} 2x,\ x\in(0,1/2)\\ -2x+2,\ x\in(1/2,1)\end{array}\right\}\\ \phi_3 &= 2x-1,\ x\in(1/2,1)\end{aligned}$$

Now, we have that the space for our discrete variational method is the span of these basis functions. So $u = a_1\phi_1 + a_2\phi_2 + a_3\phi_3\phi_4$. It is these a that we would attempt to solve. The stiffness matrix could now be formed by $(\phi_i', \phi_j') = \int_0^1 \phi_i'\phi_j'\,dx$. The solution to the full system $Ax = b$ with

$$A_{ij} = \left(\phi_i', \phi_j'\right) + 2\left(\phi_i, \phi_j\right)$$

(WITH the mass matrix) would give coefficients a_1, a_2 and a_3 so that $u = a_1\phi_1 + a_2\phi_2 + a_3\phi_3$ approximates the solution to the BVP ODE. To actually find A_{ij}, just integrate over the support of the basis fuctions. For example, the stiffness matrix part of A would be:

$$\begin{aligned}A_{11} &= \int_0^{1/2} -2\cdot -2\,dx,\\ A_{12} &= \int_0^{1/2} -2\cdot 2\,dx,\\ A_{22} &= \int_0^{1/2} 2\cdot 2\,dx + \int_{1/2}^{1} -2\cdot -2\,dx,\\ &\vdots\end{aligned}$$

January 2005

1.3 Define a multi-step method for approximately solving ODEs $y' = f(t, y)$ using (i'm rewriting the below differently than stated)

$$y_n = y_{n-3} + \frac{3h}{8}\left[f_n + 3f_{n-1} + 3f_{n-2} + f_{n-3}\right]$$

(a) Show this method is of order exactly 4. (b) Is this method 0-stable? Strongly stable? (c) Is $z = h\lambda = -8/3$ in the region of absolute stability fo this method?

Here we go again with the multistep method. We rewrite this as

$$y_n - y_{n-3} = h\left[\frac{3}{8}f_n + \frac{9}{8}f_{n-1} + \frac{9}{8}f_{n-2} + \frac{3}{8}f_{n-3}\right]$$

and I define (from lowest to highest terms) $a_0 = -1$, $a_1 = 0$, $a_2 = 0$ and $a_3 = 1$. Then $b_0 = \frac{3}{8}$, $b_1 = \frac{9}{8}$, $b_2 = \frac{9}{8}$ and $b_3 = \frac{3}{8}$. Then go through the orders:

$$\begin{aligned}0:&\ \sum_{i=0}^{3} a_i = -1+1 = 0\\ 1:&\ \sum_{i=0}^{3}\left(ia_i - b_i\right) = 3 - \tfrac{3}{8} - \tfrac{9}{8} - \tfrac{9}{8} - \tfrac{3}{8} = 0\\ 2:&\ \sum_{i=0}^{3}\left(\frac{1}{2!}i^2a_i - ib_i\right) = \frac{9}{2} - \frac{9}{8} - \frac{18}{8} - \frac{9}{8} = 0\\ 3:&\ \sum_{i=0}^{3}\left(\frac{1}{3!}i^3a_i - \frac{1}{2!}i^2b_i\right) = \frac{9}{2} - \frac{9}{16} - \frac{18}{8} - \frac{9}{2}\frac{3}{8} = 0\\ 4:&\ \sum_{i=0}^{3}\left(\frac{1}{4!}i^4a_i - \frac{1}{3!}i^3b_i\right) = \frac{27}{8} - \frac{1}{6}\frac{9}{8} - \frac{4}{3}\frac{9}{8} - \frac{9}{2}\frac{3}{8} = 0\\ 5:&\ \sum_{i=0}^{3}\left(\frac{1}{5!}i^5a_i - \frac{1}{4!}i^4b_i\right) = \frac{243}{120} - \frac{1}{24}\frac{9}{8} - \frac{2}{3}\frac{9}{8} - \frac{27}{8}\frac{3}{8} = -\frac{3}{80} \neq 0\end{aligned}$$

and find this is 4^{th}order (exactly).

As for stability, forming a polynomial out of the LHS, we have

$$p(z) = z^3 - 1 = (z-1)(z^2+z+1)$$

which has roots

$$z = 1, \frac{-1 \pm i\sqrt{3}}{2}.$$

Since $\left|\frac{-1\pm i\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$ we have three simple roots of modulous 1, thus this method is 0-stable but NOT strongly stable (it would have to have only $z=1$ as a root of mod 1 to be strongly stable).

A-stability can be found by also writing the RHS as

$$q(z) = \frac{3}{8}z^3 + \frac{9}{8}z^2 + \frac{9}{8}z + \frac{3}{8}$$

then finding the set of complex numbers λh so that the roots of $p(z) - \lambda h q(z)$ lie on the interior of the unit disk. DO NOT find the region on A-stability. Only see if $h\lambda = -8/3$ is in this region. We have

$$p - \frac{8}{3}q = z^3 - 1 + \frac{8}{3}\left(\frac{3}{8}z^3 + \frac{9}{8}z^2 + \frac{9}{8}z + \frac{3}{8}\right) = 3z^2 + 3z = 0.$$

This has roots 0 and -1. Since this has a root ON the unit disk, this method is not A-stable for this value.

1.7. Prove that the sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),\ n = 0, 1, 2, \ldots$$

with initial value $x_0 = 2$ is convergent. Determine the rate of convergence.

Again, I'm not quite sure if this is in your applied analysis or numerical analysis class, but we'll do a quick solution. Write

$$g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right) = \frac{1}{2}\left(\frac{x^2+2}{x}\right),$$

then

$$g'(x) = \frac{1}{2} - \frac{1}{x^2}.$$

Note that

$$\left|\frac{1}{2} - \frac{1}{x^2}\right| < 1$$

for $x < -\sqrt{6}/3$ and $x > \sqrt{6}/3$. Now, where are the fixed points? Well, solve

$$g(x) - x = 0 = -\frac{x^2-2}{2x} = 0$$

so $x = \pm\sqrt{2}$. Since $x_0 = 2$ and $x = \sqrt{2}$ are larger than $\sqrt{6}/3$ we have convergence by the fixed point theorem (ie $|g'(x)| < 1$ on $x > \sqrt{6}/3$ so the fixed point converges). Now, generally we have some sort of nice linear convergence, but we just happen to have the fact that

$$g'\left(\sqrt{2}\right) = \frac{1}{2} - \frac{1}{2} = 0,$$

so the derivative is zero AT the fixed point. In this case we actually see what the 2nd derivative does. We have

$$g''(x) = \frac{2}{x^3}.$$

Since this is continuous and bounded for $x > \sqrt{6}/3$ we have quadratic convergence.

1.9 Given that a SVD of a 3×3 matrix A is $A = U\Sigma V^*$ and that the singular values are $\sigma_1 = 4$, $\sigma_2 = 1$ and $\sigma_3 = 1/2$,

(a) What is the rank of A? (b) What is $\|A\|_2$? (c) What is the 2-norm condition number $cond_2(A)$? (d) If $B \in \mathbb{C}^{3\times 3}$ has rank 2, give a sharp lower bound for $\|A - B\|_2$ and a formula for the matrix B for which the lower bound is achieved.

(a) The rank is the number of nonzero singular values, which is 3 in this case. (b) This is quite easy as well, since $\|A\|_2 = \sqrt{\lambda_{max}(A^*A)} = \max \sigma_i = 4$. (c) We have

$$cond_2(A) = \|A\|_2 \left\|A^{-1}\right\|_2 = \frac{\max \sigma_i}{\min \sigma_i} = \frac{4}{1/2} = 8.$$

This is since $AA^* = U\Sigma U^*$ then $(AA^*)^{-1} = U\Sigma^{-1}U^*$ and hence

$$\left\|A^{-1}\right\|_2 = \sqrt{\lambda_{max}\left((AA^*)^{-1}\right)} = \sqrt{\lambda_{max}(\Sigma^{-1})} = \frac{1}{\sqrt{\lambda_{min}(\Sigma)}}.$$

(d) I believe there are a few ways to do this, but none are easy if you don't actually know the lower bound already. What you want to use is

$$\|A - B\|_2 \geq \|\Sigma - \Sigma_B\|_2 \geq \min \sigma_i = 1/2.$$

Likely the best way to show this is to use the ideas in Trefethen and Bau and write A as a sum of rank 1 operators. This is not something I could whip up on the fly by the way, so perhaps there is a trick I'm missing (I'd try writing as vector norms, then using inner product properties and see if aynthing fun happens). To show this is sharp, just consider B with singular values 4 and 1. For example, take

$$B = U \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^*$$

(assuming $\Sigma = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$). Then

$$\|A - B\|_2 = \|U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} V^*\|_2 = \frac{1}{2}.$$

2.2 Let I be the two by two identity matrix. A real matrix, A, is a square root of I if $A^2 = I$. Thus $A = I$ and $A = -I$ are square roots of I. Each of these is a symmetric matrix.

(a) Find infinitely many other symmetric square roots of I. (b) For each of these, find its eigenvalues and eigenvectors. (c) Describe the geometric significance of these matrices.

We want matrices A with $A^2 = I$, or $A^2 - I = 0$. Since this is 2×2 lets just do this out:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So $a_{11}^2 + a_{12}a_{21} = 1$ implies $a_{11} = \pm(1 - a_{12}a_{21})$. Then

$$a_{11}a_{12} + a_{12}a_{22} = \pm a_{12}(1 - a_{12}a_{21}) + a_{12}a_{22} = 0$$

which implies $a_{12} = 0$ or $a_{22} \pm (1 - a_{12}a_{21}) = 0$. If $a_{12} = 0$ this implies $a_{22} = \pm 1$ and $a_{11} = \pm 1$. Then the last equation to solve would be $a_{21}a_{11} + a_{22}a_{21} = a_{21}(a_{11} + a_{22})$ which , if a_{11} and a_{22} are opposite signs, implies $a_{21} = C$. If they are the same sign, this implies $a_{21} = 0$ which gives us back the scaled identities. So one possibility is

$$\begin{pmatrix} \pm 1 & 0 \\ C & \mp 1 \end{pmatrix}.$$

The other possibility comes by symmetry and we have

$$\begin{pmatrix} \pm 1 & C \\ 0 & \mp 1 \end{pmatrix}.$$

For eigenvalues/vectors we'll do only one matrix since the other is the same. It is clear eigenvalues are ± 1. For eigenvectors, we have

$$\begin{pmatrix} 1 & 0 \\ C & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ C & -1 \end{pmatrix} \begin{pmatrix} 1 \\ C/2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ C/2 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 0 \\ C & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ C & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -C/2 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -C/2 \end{pmatrix}.$$

Similarly for the others.

As for geometric significance, well, what do these do? Take a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and apply one of the matrices

$$\begin{pmatrix} \pm 1 & 0 \\ C & \mp 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \pm x \\ Cx \mp y \end{pmatrix}.$$

So, we flip either x or y and we shift the y component by a scaled amount in x (ie Cx) . Similarly for the other way around. I'm not sure what else there is to say.

2.4 Add comments to the Matlab code...

This one returns two matrices given an $m \times n$ matrix A, W which it appears is triangular (since $W(k:m,k)$ is only assigned), and a matrix R. Not really sure what to say on this one, other than an alarm should go off in your head in the 2nd to last line here that this is a Householder QR factorization if not long before that.

2.8 Consider an $N \times N$ complex matrix A such that $A^4 = I$.

(a) What can you say about the possible eigenvalues of A? (b) Is A necessarily diagonalizable? (c) Show that $P = (I + A + A^2 + A^3)/4$ is a projection. (d) Describe the range of P.

(a) Suppose $Ax = \lambda x$. Then $x = A^4x = A^3Ax = A^3\lambda x = \lambda A^3 x = \lambda^2 A^2 x = \ldots = \lambda^4 x$. Thus if λ is an eigenvalue of A then $\lambda^4 = 1$ which means $\lambda = \pm 1, \pm i$.

(b) A is diagonalizable if it is similar to a diagonal matrix. There are many other equivalent properties, such as having unique eigenvalues. For this one, I think what we'd have to do is show that the rank of the eigenspace is N. I feel this one is diagonalizable, but I really don't see a direct way to do this. Perhaps we can try to use an idea similar to the projection problem from August 2006 (1.9 I believe). We can at least show A^2 is diagonalizable, since any $v \in \mathbb{C}^n$ can be written as $v = \frac{1}{2}\left(v + A^2 v\right) + \frac{1}{2}\left(v - A^2 v\right)$ where this first part is in the span of the eigenspace corresponding to $\lambda = 1$ and the second to $\lambda = -1$. Perhaps this can be extended to A. Alternatively, write in Jordan normal form and use that $J^4 = I$.

(c) To show a projection matrix we just need $P^2 = P$. Let us go term by term multiplying P by P

$$\begin{aligned} P\frac{I}{16} &= (I + A + A^2 + A^3)/16 \\ P\frac{A}{16} &= (A + A^2 + A^3 + I)/16 \\ P\frac{A^2}{16} &= (A^2 + A^3 + I + A)/16 \\ P\frac{A^3}{16} &= (A^2 + I + A + A^2)/16. \end{aligned}$$

Each of these is the same, so adding and dividing by 4 gives us $P^2 = P$.

(d) Since $P = (I + A + A^2 + A^3)/4$ we have $P(I - A) = 0$ so the null space of P consists of vectors of the form $x - Ax$. The range is the complement.

August 2004

1.8. Define a sequence $\{a_n\}$ by

$$a_1 = 1,\ a_{n+1} = \frac{1}{1 + a_n}.$$

Show that $\lim_{n\to\infty} a_n$ exists, and find it.

Fixed point theorem? $g(x) = \dfrac{1}{1+x}$ and $g'(x) = \dfrac{-1}{(1+x)^2}$. $x = g(x)$ implies

$$x = \frac{1}{1+x} \to x^2 + x - 1 = 0 \to x = -\frac{1 \pm \sqrt{5}}{2}.$$

Since for $x > 0$ we have $|g'(x)| < 1$ we can apply the FP theorem and the iteration converges to the positive root.

1.10. (a) Use the FEM with pw linear hat functions over a uniform mesh to discretize:

$$-\frac{d}{dx}\left((1+x)\frac{du}{dx}\right) + u = x^2 - x,\ u(0) = u(1) = 0$$

in the form $Mv = f$. Your answer must specify the values of the stiffness matrix entries m_{ij} and the load vector entries f_j as definite integrals in terms of your hat functions, and give a formula defining the hat functions.

(b) Evaluate m_{ij} explicitly in the case where there are two interior mesh points at $x = 1/3$ and $x = 2/3$. (Warning, the numbers are not pretty)

There has only been one question like this in recent years, so we'll spend a bit of time doing this. First, get the variational formulation. We multiply by v and integrate over $(0, 1)$ to get

$$\int_0^1 -u''v - u'v - xu''v + uv\,dx = \int_0^1 \left(x^2 - x\right)v.$$

Integrate by parts at lot and use the fact that $u(0) = u(1) = 0$ (and hence, we will choose v in a space so that $v(0) = v(1) = 0$):

$$\int_0^1 -u''v\,dx = \int_0^1 u'v'\,dx,\ \int_0^1 -u'' \cdot xv\,dx = \int_0^1 u'\left(v + xv'\right)\,dx.$$

Putting these in, canceling and collecting terms, we have

$$a(u, v) = \int_0^1 (1+x)\,u'v'\,dx = b(v) = \int_0^1 \left(x^2 - x\right)v.$$

The left hand side will define the stiffness matrix and the right hand side will determine the load vector.

We discretize the problem and create basis functions ϕ_i. We now wish to solve the problem: find $u \in V_0 = \text{span}\,\{\phi_i\}$so that

$$a(u, v) = b(v)\ \forall\ v \in V_0.$$

If $u = \sum \alpha_i\phi_i$ then we end up solving the system

$$M\alpha = f$$

where

$$m_{ij} = a\left(\phi_i, \phi_j\right),\ f_j = b\left(\phi_j\right).$$

If we wish to do this with piecewise linear basis function, then we simply need to define ϕ_i as such. I'd use a Lagrange basis and restrict the support of each function.

(b) I'm going to do this for a general case. Since you have 0 boundary conditions, there should be a nice simplification, but we won't use it. We have two interior points, so this will result in 6 piecewise linears. Two form each of the basis functions for the end points, then the other four form the hat functions for the interior points. Use Lagrange basis functions, so $\psi_i = \dfrac{x - x_i}{x_k - x_i}$where x_k is the neigboring point. We have functions define by:

$$\begin{aligned}
\text{on } (0, 1/3): &\quad \psi_1 = -3x + 1,\ \psi_2 = 3x \\
\text{on } (1/3, 2/3): &\quad \psi_3 = -3x + 2,\ \psi_4 = 3x - 1 \\
\text{on } (2/3, 1): &\quad \psi_5 = -3x + 3,\ \psi_6 = 3x - 2.
\end{aligned}$$

So, the idea is we now have 4 basis functions from these 6 piecewise linears:

$$\begin{aligned}
\phi_1 &= \psi_1 = -3x + 1,\ \ x \in (0, 1/3) \\
\phi_2 &= \psi_2 \cup \psi_3 = \left\{ \begin{array}{c} 3x,\ \ x \in (0, 1/3) \\ -3x + 2,\ \ x \in (1/3, 2/3) \end{array} \right\} \\
\phi_3 &= \psi_4 \cup \psi_5 = \left\{ \begin{array}{c} 3x - 1,\ \ x \in (1/3, 2/3) \\ -3x + 3,\ \ x \in (2/3, 1) \end{array} \right\} \\
\phi_4 &= \psi_6 = 3x - 2,\ \ x \in (2/3, 1)
\end{aligned}$$

Now, we define a 4×4 tridiagonal stiffness matrix (tridiagonal because only neighboring basis functions will have nonempty support) through

$$\sum_{i,j=1}^{4} \int_0^1 (1+x)\, \phi_i' \phi_j'\, dx.$$

Just to do a couple rows, we have

$$\begin{aligned}
m_{11} &= \int_0^{1/3} (1+x) \cdot -3 \cdot -3\, dx = \frac{7}{2} \\
m_{12} &= \int_0^{1/3} (1+x) \cdot -3 \cdot 3\, dx = -\frac{7}{2} \\
m_{21} &= \int_0^{1/3} (1+x) \cdot 3 \cdot -3\, dx = -\frac{7}{2} \\
m_{22} &= \int_0^{1/3} (1+x) \cdot 3 \cdot 3\, dx + \int_{1/2}^{2/3} (1+x) \cdot -3 \cdot -3\, dx = -1 \\
m_{23} &= \int_{1/2}^{2/3} (1+x) \cdot -3 \cdot 3\, dx = -\frac{9}{2} \\
\vdots &= \ddots
\end{aligned}$$

NOTE: Since your boundary conditions are $u(0) = u(1) = 0$ in this case, and you're using piecewise linear elements on this given grid, you should be able to actually reduce this to a smaller system. The idea is that you must have $u(0) = u(1) = 0$ so that your solution for ϕ_1 and ϕ_4 must be zero. (Sketch out what the basis functions look like... how else are you going to get a linear combination resulting in zero on the ends?).

1.11 A linear operator $A : \mathbb{C}^m \to \mathbb{C}^m$ is defined by $A = T + 3I + T^{-1}$, where I is the identity map, and T is the cyclic shift operator

$$T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} x_m \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}.$$

Assuming N is even, what is the condition number $\kappa_2(A)$?

Well, I'd guess N means m in this case. So we need the condition number in the 2 norm. It says to recall

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2, \ \|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

You might be able to do this one directly, so compute Ax by

$$Ax = Tx + 3I + T^{-1}x = \begin{pmatrix} x_m \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} + 3 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} + \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_1 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 + x_m \\ 3x_2 + x_3 + x_1 \\ \vdots \\ 3x_m + x_1 + x_{m-1} \end{pmatrix}.$$

Then defining $x_{m+1} = x_1$ and $x_0 = x_m$ we can write:

$$\|Ax\|_2^2 = \sum_{j=1}^{m} (3x_j + x_{j+1} + x_{j-1})^2.$$

I guess you can try some sort of maximizing or something like that. I don't think I could pull it off, so I'd go with a different approach.

Why not just find T? It's a nice matrix that just moves each element down while T^{-1} moves everything up, so they are:

$$T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \ T^{-1} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 3 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

Note that A is symmetric and positive definite. Some of you may know how to get eigenvalues straight from here, maybe using an applied analysis type argument straight from the operator even, but I'm going to use a little trick. We use Geshgorin circles to estimate the eigenvalues. This says that since the matrix is diagonally dominant, then for each eigenvalue

$$\lambda \in \left[a_{ii} - \sum_{j \neq i} |a_{ij}|,\, a_{ii} + \sum_{i \neq j} |a_{ij}| \right].$$

This says $\lambda \in [1, 5]$ for our case. Further, it is clear we have eigenvectors

$$\begin{pmatrix} 3 & 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 3 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 3 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix}$$

so $\lambda_{max} = \rho(A) = 5$ and $\lambda_{min} = 1$. Thus

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\lambda_{max}}{\lambda_{min}} = 5.$$

1.12 (a) State the definition of a singular value decomposition of an $m \times n$ complex matrix. (b) Explain the relation between the singular values of a matrix and its 2-norm condition number. (c) Give an example of a poorly conditioned matrix all of whose eigenvalues are equal to one. Explain how you know that your example works. (d) Is there an $m \times m$ matrix whose singular values are all equal to one with eigenvalues λ_j such that for some j, $|\lambda_j| \neq 1$? If so, give an example. If not, prove that it is not possible.

(a) SVD: $A = U\Sigma V^*$ with, U, V unitary (orthonormal basis) and Σ is the singular values (like eigenvalues, with $Av = \sigma u$ and $A^*u = \sigma v$, clearly from $U^*A = \Sigma V^*$ and $AV = U\Sigma$). You should write more here, but we've already done plenty of SVD.

(b) $\|A\|^2 = \rho(A^*A) = \rho(V\Sigma^*U^*U\Sigma V^*) = \rho(\Sigma^*\Sigma) = \max_i \sigma_i^2$. Then similarly for A^{-1}. Then $\kappa_2(A) = \dfrac{\max|\sigma_i|}{\min|\sigma_i|}$.

(c) I don't think I'd even bother using part (a) and (b). What makes a matrix poorly conditioned? Usually a poor ratio between it's min and max terms. So consider

$$A = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix},\ A^{-1} = \begin{pmatrix} 1 & -N \\ 0 & 1 \end{pmatrix}.$$

If we use a simple L^∞ or L^1norm this is clearly ill conditioned. You can use the L^2 norm, but that requires a little more work.

(d) I believe this is not possible. Use the spectral theorem. I think you'll get $\lambda_i = \sigma_i^2$.

2.7 (a) Show that the 2−step method of the form

$$y_n = -4y_{n-1} + 5y_{n-2} + 4hf_{n-1} + 2hf_{n-2}$$

is consistent of order exactly 3. (b) Why should this method never be used?

Part (a) should be automatic by now. Why should this not be used? Well, do some stability analysis: $p(z) = z^2 - 4z + 5 = (z + 2 + i)(z + 2 - i)$ which will have roots of modulous greater than 1, so it is unstable.

2.8 Add comments to the first code fragment...

It's an adaptive Runge Kutta (RK45).

January 2004

1.6 Let

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and define

$$T(a) = aM + (1-a)I$$

where I is the identity matrix and a is a real number. (i) For what values of a is the matrix $T(a)$ non-negative? (ii) For the values of a you found in part (i), find a symmetric real 3×3 matrix $R(a)$ such that $R(a)^2 = T(a)$.

This problem is kinda interesting, so I'll go ahead and do it. We have

$$T(a) = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}.$$

So, my first answer $a \geq 0$, since the definition of a non-negative matrix is that all entries are positive. My guess is what they mean is a semi-positive definite matrix (they really seem to abuse terminology on these qualifiers),

$$x^T T(a) x \geq 0 \ \forall \ x \in \mathbb{C}^3.$$

Well, $T(a)$ is symmetric. It is positive-definite if it's eigenvalues are positive. So lets just find the eigenvalues, we get an eigenvalue equation that factors nicely and get

$$\lambda = 2a+1, 1-a, 1-a.$$

So we need $1-a \geq 0$ and $2a+1 \geq 0$. The first implies $a \leq 1$ and the second implies $a \geq -\frac{1}{2}$, so $-\frac{1}{2} \leq a \leq 1$.

(ii) You can try to do this by finding R^2 for a general symmetric R, but I think you can think of something better. Since T is symmetric, this is essentially asking for the Cholesky decomposition. The easy way to do this is to just get the diagonalization of T. Since you already have eigenvalues, try to find the eigenvectors. Pretty easy to see in this case:

$$\begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (2a+1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = (1-a) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = (1-a) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We want $A = VDV^{-1}$ with

$$V = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2a+1 & 0 & 0 \\ 0 & 1-a & 0 \\ 0 & 0 & 1-a \end{bmatrix}$$

so we need to compute V^{-1}. Pretty easy again in this case, we get

$$V^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Then $R = VD^{1/2}V^{-1}$.

1.7. Let A be the $n \times n$ matrix of the form

$$A = \begin{bmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \ddots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Find the inverse of A.

Well, it does just say to find the inverse, not prove it. I might just show what the inverse is for 2×2 up to 4×4 and jump to $n\times n$ from that. At least that's what I'd do to start to make sure I knew what I was trying to prove. The good news is that you know the inverse of A must be upper triangular and have 1's on the diagonal. Start with that:

$$AA^{-1}=\begin{bmatrix}1&1&\cdots&0&0\\0&1&\ddots&0&0\\0&0&\ddots&1&0\\\vdots&\vdots&\ddots&1&1\\0&0&\cdots&0&1\end{bmatrix}\begin{bmatrix}1&b_{12}&\cdots&b_{1,n-1}&b_{1,n}\\0&1&\ddots&\vdots&\vdots\\0&0&\ddots&b_{n-2,n-1}&\vdots\\\vdots&\vdots&\ddots&1&b_{n-1,n}\\0&0&\cdots&0&1\end{bmatrix}=I.$$

We have that $b_{i,i+1}=-1$ from the i^{th} row times the $i+1^{st}$column (for example, 1st row times 2nd column gives $b_{12}+1=0$, etc). Then work your way up from that

$$AA^{-1}=\begin{bmatrix}1&1&\cdots&0&0\\0&1&\ddots&0&0\\0&0&\ddots&1&0\\\vdots&\vdots&\ddots&1&1\\0&0&\cdots&0&1\end{bmatrix}\begin{bmatrix}1&-1&b_{1,3}&0&\cdots&b_{1,n}\\0&1&-1&b_{2,4}&\cdots&b_{2,n}\\0&0&1&-1&\ddots&\vdots\\\vdots&\ddots&\ddots&1&\ddots&\\0&0&0&0&\ddots&-1\\0&0&0&0&0&1\end{bmatrix}=I.$$

By the same argument, using the i^{th} row times the $i+2^{nd}$ column we have $b_{i,i+2}=1$, and so on.

1.9 Find constant A,B,C and D and the maximum degree of the polynomial $f(x)$ such that

(a)

$$Af(-h)+Bf(0)+Cf(h)+Df'(h)=\int_{-h}^{h}f(x)\,dx.$$

(b) What is the maximum degree of the polynomial $f(x)$ for which the above quadrature is exact?

I would use the method of undetermined coefficients. Consider $f(x)=1,x,x^2,x^3,$ etc.

$$\begin{aligned}f&=1:\ A+B+C=2h\\f&=x:\ -hA+hC+D=0\\f&=x^2:\ h^2A+h^2C+2hD=2h^3/3\\f&=x^3:\ -h^3A+h^3C+3h^2D=0\\f&=x^4:\ h^4A+h^4C+4h^3D=2h^5/5\\f&=x^5:\ -h^5A+h^5C+5h^4D=0\end{aligned}$$

We have 4 unknowns, we we'll use the first four equations. Let us take the 2nd and 4th equations. We multiply the 2nd by h^2 and subtract it from the 4^{th} to get $D=0$. Then $A=C$. The third equation gives us

$$2h^2A=2h^3/3$$

which would imply that $A=h/3$. Then $C=h/3$ and the first equation would give us $B=4h/3$. So

$$\frac{h}{3}f(-h)+\frac{4h}{3}f(0)+\frac{h}{3}f(h)=\int_{-h}^{h}f(x)\,dx.$$

Note that with these coefficients, the result does not hold for $f(x)=x^4$, thus this method is accurate only to x^4. As for the maximum degree, we have fixed nodes, so I'd say for third degree like we just found.

1.12. Semi-discretize (in space) the PDE $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$ on the interval $[-\pi,\pi]$with 2π periodic boundary conditions using 2^{nd} order central differences in the x variable on a uniform grid to get a system of N ODEs of the form $\dfrac{d\overrightarrow{u}}{dt}=A\overrightarrow{u}$ where $\overrightarrow{u}=(u_0,u_1,\ldots,u_{N-1})^T$. You may assume N even.

(a) What is A? (b) Finish discretizing the equation using forward differences in t with step size Δt. (c) For what values of Δt, in terms of Δx, will this produce a scheme so that $\overrightarrow{u}(t_m)$ remains bounded for all time? Again assume N is even.

Discretize using 2^{nd} order:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{\partial u_i}{\partial t} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

where it is assume u is a function of t. As for boundary conditions, we use periodic, which we will mean to imply that the point to the left of u_0 is u_{N-1} and the point to the right of u_{N-1} is u_0. This gives us

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \ddots & \vdots & 0 \\ 0 & 1 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & -2 & 1 & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

In time, we do a forward discretization, denoting $u_i^n = u(t_n, x_i)$ and have

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}.$$

For the last part, this is asking for the CFL condition on this finite difference discretization. The idea is that we have a time dependent DE and this should not blow up in time. We rewrite as

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) = \left(1 - \frac{2\Delta t}{\Delta x^2}\right) u_i^n + \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n + u_{i-1}^n\right).$$

Now, there are shortcuts (which I will show) but the idea is we want to do von Neumann stability analysis. So we consider a single Fourier mode $U(x,t) = \hat{U}(t)e^{ikx}$ and substitute this into our scheme. We have

$$\begin{aligned} \hat{U}^{n+1} e^{ikx_i} &= \hat{U}^n e^{ikx_i}\left(1 - 2\frac{\Delta t}{\Delta x^2} + \frac{\Delta t}{\Delta x^2} e^{-ik\Delta x} + \frac{\Delta t}{\Delta x^2} e^{ik\Delta x}\right) \\ \hat{U}^{n+1} &= \hat{U}^n\left(1 - 2\frac{\Delta t}{\Delta x^2}\left(1 - \cos k\Delta x\right)\right) = \hat{U}^n\left(1 - 4\frac{\Delta t}{\Delta x^2}\sin^2\left(k\Delta x/2\right)\right). \end{aligned}$$

Taking the max of $\sin^2$ (which is 1 of course) we have need to have

$$\left|1 - 4\frac{\Delta t}{\Delta x^2}\right| < 1 \rightarrow \frac{\Delta t}{\Delta x^2} < \frac{1}{2}$$

which implies $\Delta t < \frac{1}{2}\Delta x^2$. The 'easy' way to do this one is to take the original iteration

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) = \left(1 - \frac{2\Delta t}{\Delta x^2}\right) u_i^n + \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n + u_{i-1}^n\right),$$

slap some abs values on it, and call $u_{max}^n = \max_i u_i^n$. The write

$$\left|u_i^{n+1}\right| \leq \left(\left|1 - \frac{2\Delta t}{\Delta x^2}\right| + \frac{2\Delta t}{\Delta x^2}\right) u_{max}^n.$$

So we need

$$\left|1 - \frac{2\Delta t}{\Delta x^2}\right| + \frac{2\Delta t}{\Delta x^2} \leq 1$$

which is true as long as

$$\frac{2\Delta t}{\Delta x^2} < 1.$$

2.3. Does there exist a square matrix with all eigenvalues equal to zero and all entries not equal to zero? If so, give a specific example, if not justify why it cannot exist.

If all eigenvalues are equal to zero then we have a nilpotent matrix, so there exists a k so that $A^k = 0$. The simple example would be

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

We have

$$\det(A-\lambda I) = (1-\lambda)(-1-\lambda)+1 = \lambda^2 - 1 + 1 = \lambda^2 = 0$$

and $A^2 = 0$.

2.4 Consider

$$u'' + 4u = 1,\ u(0) = u(1) = 0,\ 0 \le x \le 1,$$

where $u = u(x)$. For this BVP: (a) Find the exact solution, (b) Derive a linear algebraic system that approximates the solution using pw linear hat function with support of width $2h$, where h is the uniform grid spacing. (c) Derive an upper bound on h that guarantees that the system constructed in part (b) will have a solution.

(a) Does $u = \frac{1}{4}\left(\frac{\cos 2 - 1}{\sin 2}\sin 2x - \cos 2x + 1\right)$ sound about right? (b) This would seem like another FEM problem to me. Discrete variational formulation of: find $u \in V_0$ such that

$$a(u,v) = -\int_0^1 u'v'\,dx + 4\int_0^1 uv\,dx = \int_0^1 v\,dx = b(u)\ \forall\ v \in V_0$$

where $V_0 = \text{span}\{\phi_i\}_{i=1}^N$ are the pw linear hat basis functions. The hat part of the basis functions will look something like Lagrange polynomials, where on $[x_{i-1}, x_{i+1}]$ we'll have basis functions making up the hat that look like

$$\frac{x - x_{i-1}}{h}, \frac{x_{i+1} - x}{h}$$

on (x_{i-1}, x_i) and (x_i, x_{i+1}), respectively. (c) I actually don't know what they're getting at here. The matrix may be ill conditioned for large h, but it may be for small h as well. Lets try to just look at some interior row. Let ϕ_j be some interior hat function with support on $[x_{j-1}, x_{j+1}]$. Then

$$\begin{aligned} a(\phi_j,\phi_j) &= -\int_{x_{j-1}}^{x_j} \frac{1}{h^2}\,dx - \int_{x_j}^{x_{j+1}} \frac{1}{h^2}\,dx + 4\int_{x_{j-1}}^{x_j}\left(\frac{x-x_{j-1}}{h}\right)^2 dx + 4\int_{x_j}^{x_{j+1}}\left(\frac{x_{j+1}-x}{h}\right)^2 dx \\ &= -\frac{2}{h} + \frac{8}{3}h \end{aligned}$$

and

$$\begin{aligned} a(\phi_{j+1},\phi_j) &= \int_{x_j}^{x_{j+1}} \frac{1}{h^2}\,dx + 4\int_{x_j}^{x_{j+1}}\left(\frac{x_{j+1}-x}{h}\right)\left(\frac{x-x_j}{h}\right) dx \\ &= \frac{2}{h} + \frac{2}{3}h. \end{aligned}$$

with $a(\phi_{j-1},\phi_j) = a(\phi_{j+1},\phi_j)$ by a symmetry argument. Now what, well, I don't know. Check your notes and see if they do anything that resembles this.

2.6 For $A = \begin{pmatrix} 10^{-2} & -1 \\ 1 & 10^{-1} \end{pmatrix}$. (a) Find the LU factorization of A, namely $A = LU$. (b) Find the $PA = L'U'$ factorization based on partial pivoting. (c) Use both $A = LU$ and $PA = L'U'$ with forwards and backwards substitution to solve $Ax = b = (1,1)^T$ using arithmetic with 3 significant decimal digits of relative accuracy.

Well, it's not symmetric, so we use a Doolittle or Crout factorization:

$$\begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix}.$$

So we get

$$L = \begin{pmatrix} 1 & 0 \\ 10^2 & 1 \end{pmatrix}, U = \begin{pmatrix} 10^{-2} & -1 \\ 0 & 10^2 + 10^{-1} \end{pmatrix}.$$

If we pivot, then

$$A = \begin{pmatrix} 1 & 10^{-1} \\ 10^{-2} & -1 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 10^{-2} & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 10^{-1} \\ 0 & -1 - 10^{-3} \end{pmatrix}.$$

We're solve for only LU, since the other is similar. Recall that we solve in steps.

$$LUx = b = Ly = b$$

so we solve for y first.

$$\begin{pmatrix} 1 & 0 \\ 10^2 & 1 \end{pmatrix} y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

implies $y_1 = 1$, $y_2 = 1 - 10^2 = -99$ which will be exact. Then we want to solve $Ux = y$.

$$\begin{pmatrix} 10^{-2} & -1 \\ 0 & 10^2 + 10^{-1} \end{pmatrix} x = \begin{pmatrix} 1 \\ -99 \end{pmatrix}$$

so $x_2 = \frac{-99}{10^2+10^{-1}} \approx -0.989$. Then $x_1 = (1 + x_2)/10^{-2} = 1.10$. You should get a more precise answer for the PLU.

General SVD Problem: Find the SVD of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problems where you need to use the SVD pop up 3 or 4 times in the last few years exams, so this is something I would know VERY well. Recall the SVD is the Singular Value Decomposition. We write the matrix $A \in \mathbb{C}^{m\times n}$ as $A = U\Sigma V^*$ (here $V^* = V^T$) where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{C}^{m\times n}$ is a diagonal matrix. This can be used as just another factorization method in proving many interesting linear algebra theorems, but it has a very important application. If $Ax = b$ is an underdetermined system, then the minimal solution (in terms of $\|Ax - b\|_2$) is given by $x = A^+b$ where $A^+ = V\Sigma^{-1}U^*$. So if we have an SVD, we can find the least squares solution very easily.

Start with $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$. This is a nice square invertible matrix so things should come out pretty easy. To find the SVD, we first find A^*A:

$$A^*A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Now we find it's eigenvalues/eigenvectors:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \pm \sqrt{2} \\ 1 \end{pmatrix} = 3 \pm 2\sqrt{2} \begin{pmatrix} 0 \\ 1 \pm \sqrt{2} \\ 1 \end{pmatrix}.$$

So we have singular values (from highest to lowest) given by the square root of these eigenvalues

$$\Sigma = \begin{pmatrix} 1+\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2}-1 \end{pmatrix}.$$

The corresponding normalized eigenvectors become our V matrix:

$$V = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & 0 & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & 0 & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}.$$

We then compute $u_i = \sigma_i^{-1} A v_i$ for each column of V to get U. For example, the first row of U is:

$$u_1 = \frac{1}{1+\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} = \frac{1}{1+\sqrt{2}} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{3+2\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}.$$

These numbers seem to get messy, but that's the idea.

Let $A = \begin{pmatrix} 0 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This is not square, so the process for finding the SVD will be a little different then a nice eigenvalue decomposition like in the last problem. So we write

$$A^*A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And have the eigenvalues/vectors

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This is really nice, since we have singular values $2, 1$ and 0 forming

$$\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with corresponding eigenvectors forming

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$u_1 = \frac{1}{2} A v_1 = \begin{pmatrix} -0.8 \\ 0.6 \\ 0 \\ 0 \end{pmatrix}, u_2 = A v_2 = \begin{pmatrix} 0.6 \\ 0.8 \\ 0 \\ 0 \end{pmatrix}.$$

Since the other singular values are zero, we can choose u_3 and $u_4$however we want (so that U is unitary!). The easy choice is $u_3 = e_3$ and $u_4 = e_4$. So we get

$$A = \begin{pmatrix} -0.8 & 0.6 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Multistep: Check consistency, stability, A-stablility (if possible) and derive an expression for the local truncation error of the following multistep methods:

1. $y_{i+1} = y_{i-2} + \frac{3}{2} h \left(f_i + f_{i-1} \right)$
2. $y_{i+1} = y_{i-1} + \frac{1}{3} h \left(f_{i+1} + 4 f_i + f_{i-1} \right)$

1. There is a nice shortcut I ALWAYS use on these problems. The idea is that the multistep method can be written as a linear functional:

$$Ly = \sum_{i=0}^{k} \left(a_i y \left(ih \right) - h b_i y' \left(ih \right) \right) = 0$$

where here we assume the starting value is 0 and the method goes up to kh. We can expand this functional about 0 and get

$$Ly = d_0 y \left(0 \right) + d_1 h y' \left(0 \right) + d_2 h^2 y'' \left(0 \right) + \ldots$$

Take the Taylor series for y and y':

$$y \left(ih \right) = \sum_j \frac{(ih)^j}{j!} y^{(j)}(0), \; y' \left(ih \right) = \sum_j \frac{(ih)^j}{j!} y^{(j+1)}(0)$$

put them into Ly and collect powers of h. This gives us the d values:

$$\begin{aligned} d_0 &= \sum_{i=0}^{k} a_i \\ d_1 &= \sum_{i=0}^{k} (ia_i - b_i) \\ d_2 &= \sum_{i=0}^{k} \left(\frac{i^2}{2!} a_i - ib_i \right) \\ d_3 &= \sum_{i=0}^{k} \left(\frac{i^3}{3!} a_i - \frac{i^2}{2!} b_i \right) \\ &\vdots \quad \vdots \quad \vdots \end{aligned}$$

For our particular problem, we have four terms $i+1, i, i-1$ and $i-2$. So we write in the form:

$$\mathbf{a} \cdot \mathbf{y} = a_3 y_{i+1} + a_2 y_i + a_1 y_{i-1} + a_0 y_{i-2} = h \left(b_3 f_{i+1} + b_2 f_i + b_1 f_{i-1} + b_0 f_{i-2} \right) = h \mathbf{b} \cdot \mathbf{f}.$$

$$\mathbf{a} = (a_3, a_2, a_1, a_0) = (1, 0, 0, -1) \, , \, \mathbf{b} = (b_3, b_2, b_1, b_0) = \left(0, \frac{3}{2}, \frac{3}{2}, 0 \right).$$

We get

$$\begin{aligned} d_0 &= -1 + 1 = 0 \\ d_1 &= 3 - \frac{3}{2} - \frac{3}{2} = 0 \\ d_2 &= \frac{9}{2} - 3 - \frac{3}{2} = 0 \\ d_3 &= \frac{9}{2} - 3 - \frac{3}{4} = \frac{3}{4}. \end{aligned}$$

So this method is 2^{nd} order. By the way that we defined d, this gives us the actual truncation error term! Since

$$Ly = d_0 y(0) + d_1 h y'(0) + d_2 h^2 y''(0) + \dots$$

we have, in this case

$$Ly = d_3 h^3 y^{(3)}(0) + O\left(h^4\right).$$

Truncation error is defined as

$$\tau_{i+1} = \frac{y_{i+1} - y_{i-2}}{h} - \frac{3}{2} \left(f_i + f_{i-1} \right) = \frac{1}{h} Ly = \frac{3}{4} h^2 y^{(3)}(0) + O\left(h^3\right).$$

For stability, we write the left as a polynomial $p(z) = \sum_{i=0}^{k} a_i z^i$and similarly the right as $q(z) = \sum_{i=0}^{k} b_i z^i$. Stability means that all roots of p lie inside $|z| \leq 1$ and if a root is such that $|z| = 1$ then it must be simple. We have

$$p(z) = z^3 - 1 = 0 \rightarrow z = 1, \frac{-1 \pm i\sqrt{3}}{2}.$$

These all have modulous 1, and they are all simple roots. This method is therefore stable. In some of the notes they use the terms 0−stability and strongly stable. I believe 0-stability is what we just proved. A method is strongly stable if the only root with $|z| = 1$ is $z = 1$. (So this would not be strongly stable). As for A-stability, we let $\phi(z) = p(z) - \lambda h q(z)$. A method is A-stable if the roots of ϕ are in the interior of the unit disk for all $h > 0$ and $Re(\lambda) < 0$. For this method, we have

$$\phi(z) = z^3 - 1 - \lambda h \left(\frac{3}{2} z^2 + \frac{3}{2} z \right) = 0.$$

Now, we can't really find the roots (at least I can't from just looking at it) so the idea here would be that if they ask, you can verify that a given value is in the region of A-stability.

2. We have $y_{i+1} = y_{i-1} + \frac{1}{3}h\left(f_{i+1} + 4f_i + f_{i-1}\right)$, so

$$\mathbf{a} = (a_2, a_1, a_0) = (1, 0, -1)\,,\ \mathbf{b} = (b_2, b_1, b_0) = \left(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}\right).$$

Compute the d values:

$$\begin{aligned} d_0 &= -1 + 1 = 0 \\ d_1 &= 2 - \frac{1}{3} - \frac{4}{3} - \frac{1}{3} = 0 \\ d_2 &= 2 - \frac{2}{3} - \frac{4}{3} = 0 \\ d_3 &= \frac{4}{3} - \frac{2}{3} - \frac{2}{3} = 0 \\ d_4 &= \frac{2}{3} - \frac{4}{9} - \frac{2}{9} = 0 \\ d_5 &= \frac{4}{15} - \frac{2}{9} - \frac{1}{18} = -\frac{1}{90}. \end{aligned}$$

We have a 4^{th} order scheme with $\tau_{i+1} = -\frac{1}{90}h^4 y^{(5)}(0) + O\left(h^5\right)$.

We have

$$p(z) = z^2 - 1 = 0 \to z = \pm 1.$$

This method is therefore stable (but not strongly stable). As for A-stability, we have

$$\phi(z) = z^2 - 1 - \lambda h\left(\frac{1}{3}z^2 + \frac{4}{3}z + \frac{1}{3}\right) = 0 = z^2\left(1 - \frac{\lambda h}{3}\right) + z\left(\frac{4\lambda h}{3}\right) + \left(-1 - \frac{\lambda h}{3}\right).$$

This is not going to be A-stable. Consider $\lambda h = -3$, then

$$\phi(z) = 2z^2 + 4z = 2z\,(z+2) = 0$$

which has a root of $z = -2$.

Aug03, 2.3: Consider a system of ODEs

$$\begin{aligned} u' &= v \\ v' &= -u, \end{aligned}$$

approximated on a staggered grid by the following FD method,

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} &= v^{n+1/2} \\ \frac{v^{n+3/2} - v^{n+1/2}}{\Delta t} &= -u^{n+1}. \end{aligned}$$

Determine the values of Δt that are necessary and sufficient for stability of the above discretization.

Let us write this a little differently first:

$$\begin{aligned} u^{n+1} &= u^n + \Delta t v^{n+1/2} \\ v^{n+3/2} + \Delta t u^{n+1} &= v^{n+1/2}. \end{aligned}$$

Writing as a system, we have

$$\begin{pmatrix} 1 & 0 \\ \Delta t & 1 \end{pmatrix}\begin{pmatrix} u^{n+1} \\ v^{n+3/2} \end{pmatrix} = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^n \\ v^{n+1/2} \end{pmatrix}$$

or $AU^{n+1} = BU^n$. We get

$$U^{n+1} = A^{-1}BU^n = \begin{pmatrix} 1 & 0 \\ -\Delta t & 1 \end{pmatrix}\begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^n \\ v^{n+1/2} \end{pmatrix} = \begin{pmatrix} 1 & \Delta t \\ -\Delta t & 1 - \Delta t^2 \end{pmatrix}.$$

By stability, I assume they mean values so that U^{n+1} does not blow up. Why should it not blow up? Well $u' = v$ so $v' = u'' = -u$ which has solutions $u = a\sin(x) + b\cos(x)$ so this should not blow up. This corresponds to the matrix

having eigenvalues less than one (Do a eigen decomposition and write $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$). Computing the eigenvalues, we get

$$(1-\lambda)\left(1-\Delta t^2-\lambda\right)+\Delta t^2 = \lambda^2+\left(\Delta t^2-2\right)\lambda+1=0$$

which, using the quadratic formula, gives us

$$\lambda = -\frac{\Delta t^2}{2}+1\pm\Delta t\frac{\sqrt{\Delta t^2-4}}{2}.$$

If $\Delta t^4-4\Delta t^2<0$ or $-2<\Delta t<2$ then the eigenvalues are imaginary and have modulous 1. If $\Delta t=\pm 2$ then we have double roots which means we will not have a stable method. If $\Delta t>2$ then we need to have $|\lambda|<1$ for both roots, but

$$\lambda = 1-\frac{\Delta t^2}{2}+\Delta t\frac{\sqrt{\Delta t^2-4}}{2}$$

will be always decreasing for $\Delta t>2$, and it starts at $\lambda=-1$, which implies we must have a root with absolute value greater than one. Same goes for $\Delta t<-2$.

Jan 03 1.11: Find a relation b/t the positive parameters α and β such that the Gauss-Seidel method is convergent when applied to the linear system $Ax=b$ with matrix A of the form

$$A=\begin{pmatrix}\alpha & -\beta\\ -\alpha & \alpha\end{pmatrix}.$$

Recall Richardson, Jacobi, and Gauss-Seidel are all iteration methods based on matrix partitioning:

$$A=D-L-U=\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}-\begin{pmatrix}0 & 0\\ \alpha & 0\end{pmatrix}-\begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}.$$

The idea is that $Ax=Dx-Lx-Ux=b$. We choose some of the x values to be explicit x^n and others to be implicit x^{n+1}. The more implicit the system is, the more work is done per iteration. If each x is x^{n+1} then we are just inverting the matrix. Jacobi iteration uses only the diagonal as implicit: $Dx^{n+1}-Lx^n-Ux^n=b$ so

$$\text{Jacobi:} Dx^{n+1}=Lx^n+Ux^n+b.$$

Gauss-Seidel also uses one of the triangular blocks as implicit. Usually we choose the lower: $Dx^{n+1}-Lx^{n+1}-Ux^n=b$ so

$$\text{Gauss-Seidel: } (D-L)\,x^{n+1}=Ux^n+b.$$

The actual solution to this problem is, of course

$$x=A^{-1}b=\frac{1}{\alpha(\alpha-\beta)}\begin{pmatrix}\alpha & \beta\\ \alpha & \alpha\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix}=\frac{1}{(\alpha-\beta)}\begin{pmatrix}b_1+\frac{\beta}{\alpha}b_2\\ b_1+b_2\end{pmatrix}.$$

Gauss-Seidel for this problem results in

$$\begin{pmatrix}\alpha & 0\\ -\alpha & \alpha\end{pmatrix}x^{n+1}=\begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}x^n+b$$

or

$$\begin{aligned}x^{n+1} &= \frac{1}{\alpha}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}x^n+\frac{1}{\alpha}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}b=\frac{1}{\alpha}\begin{pmatrix}0 & \beta\\ 0 & \beta\end{pmatrix}x^n+\frac{1}{\alpha}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}b\\ &= \frac{\beta}{\alpha}\begin{pmatrix}x_2^n\\ x_2^n\end{pmatrix}+\frac{1}{\alpha}\begin{pmatrix}b_1\\ b_1+b_2\end{pmatrix}.\end{aligned}$$

In order for this to converge, we'd need $\left\|x^{n+1}-x^n\right\|\to 0$, but

$$\begin{aligned}x^{n+1}-x^n &= \frac{1}{\alpha}\begin{pmatrix}0 & \beta\\ 0 & \beta\end{pmatrix}x^n+\frac{1}{\alpha}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}b-\frac{1}{\alpha}\begin{pmatrix}0 & \beta\\ 0 & \beta\end{pmatrix}x^{n-1}-\frac{1}{\alpha}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}b\\ &= \frac{1}{\alpha}\begin{pmatrix}0 & \beta\\ 0 & \beta\end{pmatrix}\left(x^n-x^{n-1}\right).\end{aligned}$$

So we need

$$\begin{pmatrix}0 & \beta/\alpha\\ 0 & \beta/\alpha\end{pmatrix}^k=\begin{pmatrix}0 & (\beta/\alpha)^k\\ 0 & (\beta/\alpha)^k\end{pmatrix}\to 0$$

which implies $|\beta/\alpha| < 1$ or $|\beta| < |\alpha|$. Note that this implies the original matrix is strictly diagonally dominant, which is a theorem for Gauss-Seidel (strictly diagonally dominant implies convergence).

Jan 03 2.5: Consider the continuous eigenvalue problem $-\dfrac{d^2u}{dx^2} = \lambda^2 u$, on $x \in [0, \pi]$ with $u(0) = u(\pi) = 0$. The continuous problem is discretized by a centered finite difference approximation and results in the following discrete eigenvalue problem:

$$-u_{j+1} + 2u_j - u_{j-1} = \tilde{\lambda}^2 h^2 u_j, \ j = 1, \ldots (N-1),$$

with $u_0 = u_N = 0$. (a) Write down an explicit expression for the truncation error. (b) Find all eigenpairs for both the continous and discrete problems. (c) Show that $\left|\tilde{\lambda}_1 - \lambda_1\right| = O\left(h^2\right)$ as $h \to 0$.

Use Taylor series:

$$u_{j\pm1} = u_j \pm h u_j' + \frac{h^2}{2} u_j'' \pm \frac{h^3}{3!} u_j^{(3)} + \frac{h^4}{4!} u_j^{(4)} + O\left(h^5\right)$$

so

$$\frac{-u_{j+1} + 2u_j - u_{j-1}}{h^2} = -u_j'' - \frac{h^2}{12} u^{(4)}\left(\xi_j\right).$$

Our truncation error is

$$\tau(x_j) = -\frac{d^2u}{dx^2} - \lambda^2 u - \left(\frac{-u_{j+1} + 2u_j - u_{j-1}}{h^2} - \tilde{\lambda}^2 u_j\right) = \frac{h^2}{12} u^{(4)}\left(\xi_j\right) + \left(\tilde{\lambda}^2 - \lambda^2\right) u_j.$$

To find the eigenpairs for the original problem, just solve it. What is the solution to $u'' + \lambda^2 u = 0$? Isn't it $u(x) = a \sin \lambda x + b \cos \lambda x$? Then $u(0) = 0 \to b = 0$. $u(\pi) = a \sin \lambda\pi = 0$ implies $\sin \lambda\pi = 0$ so $\lambda \in \mathbb{Z}$. As for the discrete system, if I remember correctly, there is a nice little trick. We consider eigenvectors of the form $\beta \sin\left(\alpha_j l\right)$ and $\beta \cos\left(\alpha_j l\right)$, here α_j is some parameter that can vary depending on the row, β is some constant, and l is the column index. Applying the matrix to this eigenvector, for a general row, we get

$$\begin{aligned} \beta\left(2\sin\left(\alpha_j l\right) - \sin\left(\alpha_j\left(l-1\right)\right) - \sin\left(\alpha_j\left(l+1\right)\right)\right) &= \beta\left(2\sin\left(\alpha_j l\right) - 2\cos\left(\alpha_j\right)\sin\left(\alpha_j l\right)\right) \\ &= \beta \sin\left(\alpha_j l\right)\left(2 - 2\cos\left(\alpha_j\right)\right) \end{aligned}$$

and

$$\begin{aligned} \beta\left(2\cos\left(\alpha_j l\right) - \cos\left(\alpha_j\left(l-1\right)\right) - \cos\left(\alpha_j\left(l+1\right)\right)\right) &= \beta\left(2\cos\left(\alpha_j l\right) - 2\cos\left(\alpha_j\right)\cos\left(\alpha_j l\right)\right) \\ &= \beta \cos\left(\alpha_j l\right)\left(2 - 2\cos\left(\alpha_j\right)\right) \end{aligned}$$

which means the eigenvalues are $2 - 2\cos\left(\alpha_j\right)$ and the eigenvectors are linear combinations of the sinand cos term. To find α_j, we apply the boundary conditions:

$$\begin{aligned} l &= 0 \to \hat{\beta} = 0 \\ l &= N \to \beta \sin\left(\alpha_j N\right) = 0 \to \alpha_j = \frac{j\pi}{N} \end{aligned}$$

so the eigenvalues are $2 - 2\cos\left(\frac{j\pi}{N}\right)$ with eigenvectors $v_j = \beta \sin\left(\frac{j\pi}{N} l\right)$. (c) I'm thinking that 0 is still the smallest eigenvalue (maybe $u = C$?), then

$$\begin{aligned} \left|\tilde{\lambda}_1 - \lambda_1\right| &= \left|2 - 2\cos\left(\frac{\pi}{N}\right)\right| = \left|2 - 2\cos\left(\frac{\pi}{N}\right)\right| = \left|\left(\frac{\pi}{N}\right)^2 - \frac{1}{12}\left(\frac{\pi}{N}\right)^4 + O\left(h^6\right)\right| \\ &= \pi^2 h^2 + O\left(h^4\right). \end{aligned}$$